Chapter

4

Radicals

LEARNING GOALS

You will be able to develop your number sense and logical reasoning ability by

- Simplifying radical expressions
- Solving problems that involve operations on radicals with numerical and variable radicands
- Solving problems that involve radical equations

? The rocket launch of the New Horizons probe was so powerful that the probe escaped Earth's gravity. It will reach Pluto in 2015. Escape velocity, V_e, is the speed an object needs to break free from the gravitational pull of another object. At the surface of Earth, the escape velocity for an object is about 11 172 m/s. Escape velocity can be calculated using the formula

$$V_e = \sqrt{\frac{2GM}{r}}$$

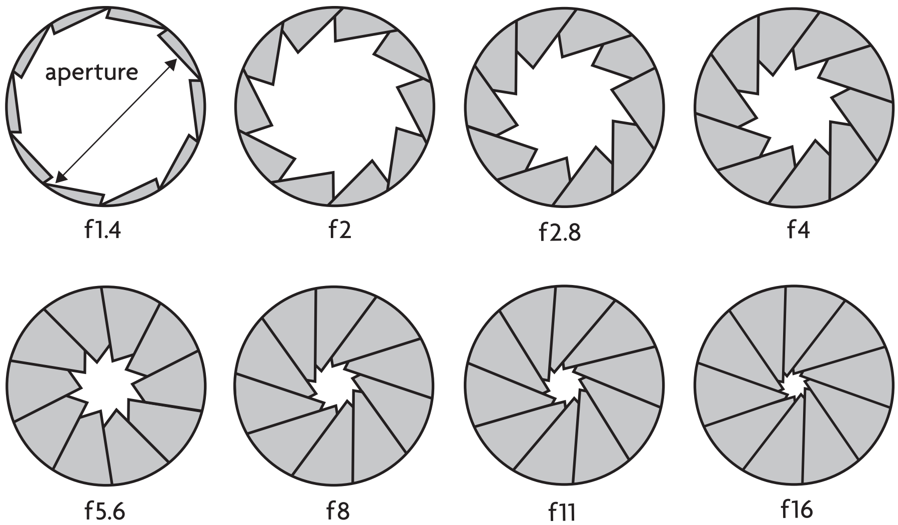
where G, the gravitational constant, is $6.67 \cdot 10^{-11} \frac{\text{Nm}^2}{\text{kg}^2}$, and r, the radius of Earth, is $6.38 \cdot 10^6$ m. How can this formula be used to determine M, the mass of Earth, in kilograms?

4 Getting Started

Photography

Carla is taking a photography course. She started using the f-stop ring on her camera lens knowing that the f-stop ring controls the aperture.

Aperture is a physical measure of how "open" a lens is. A lens opens and closes to allow more or less light to enter the camera as shown in the diagram on the next page. The aperture functions in the same way the irises of our eyes adjust to bright and dark light conditions. The f-stop number represents the ratio of the focal length of the lens, which is fixed, to the diameter of the opening, which is variable. As the diameter of the opening increases, the f-stop number decreases.



? How are the f-stop numbers on a camera determined?

A. Start with the number $\sqrt{2}$ and multiply it by $\sqrt{2}$. What **entire radical** do you get?

B. Take your answer from part A and multiply it by $\sqrt{2}$. What entire radical do you get?

C. Repeat step B five more times.

D. Write the list of entire radicals you get as a sequence of numbers in increasing order.

E. Express each entire radical in your sequence in simplest form. Which radicals were **perfect squares**? Which radicals resulted in **mixed radicals**?

F. Evaluate each number in your sequence from part D to one decimal place using a calculator.

G. Explain how the f-stop numbers are related to the values you found in part F.

WHAT DO You Think?

Decide whether you agree or disagree with each statement. Explain your decision.

1. Every real number has only one square root.
2. The radicand of a mixed or entire radical can never be a negative number.
3. Any solution to an equation obtained using inverse operations will be valid.

4.1 Mixed and Entire Radicals

YOU WILL NEED
- graph paper

GOAL

Compare and express numerical radicals in equivalent forms.

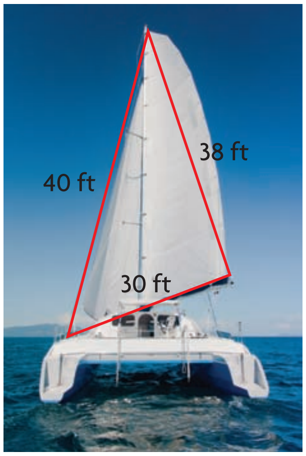
EXPLORE...
- Draw a number line from -10 to 10 and choose any two opposite numbers. Square these numbers. What do you notice? Will you always get the same result when you square two opposite numbers? What does this imply about the square root of a number?

LEARN ABOUT the Math

Sandy is designing a mainsail for a new catamaran with the dimensions shown. He needs to estimate the area of the sail, to know how much material to buy. On the Internet, Sandy found Heron's formula, which can be used to determine the area of a triangle, A:

$$A = \sqrt{p(p-a)(p-b)(p-c)}$$

where a, b, and c are the triangle's side lengths and p is half of the triangle's perimeter.



? How can Sandy estimate the area of the sail?

EXAMPLE 1 Estimating an area

Estimate the area of the mainsail using Heron's formula.

Tim's Solution

Half of the triangular sail's perimeter, p, is:

$$p = \frac{40 + 38 + 30}{2}$$

$$p = \frac{108}{2}$$

$$p = 54$$

First I determined p, the value of half the perimeter.

The area of the triangular sail, A, is:

$$A = \sqrt{p(p-a)(p-b)(p-c)}$$

$$A = \sqrt{(54)(54-40)(54-38)(54-30)}$$

$$A = \sqrt{(54)(14)(16)(24)}$$

The square root of a positive number provides two answers; the **principal square root**, and the **secondary square root**. As area cannot be negative, I know I want to determine only the principal square root.

I substituted the value of p and the lengths of the three sides into Heron's formula and simplified.

principal square root

The positive square root of a real number, x, denoted as $\sqrt{x}$; for example, the principal square root of 16 is $\sqrt{16}$, or 4.

secondary square root

The negative square root of a real number, x, denoted as $-\sqrt{x}$; for example, the secondary square root of 16 is $-\sqrt{16}$, or -4.

$$A = \sqrt{(2 \cdot 3^3)(2 \cdot 7)(2^4)(2^3 \cdot 3)}$$

I wrote each factor in terms of its prime factors to try to write the entire radical as a mixed radical with the smallest radicand. This would make estimation easier.

$$A = \sqrt{2^9 \cdot 3^4 \cdot 7}$$

I used the exponent laws to combine like bases.

$$A = \sqrt{2^8 \cdot 3^4} \cdot \sqrt{2 \cdot 7}$$

Any power squared results in a power whose exponent doubles, $(a^n)^2 = a^{2n}$. So, to determine the square root of a power, the exponents must be divisible by 2 or be even. I rearranged the factors into two products, one of which had the largest even exponents possible.

$$A = 2^4 \cdot 3^2 \sqrt{14}$$
$$A = 16 \cdot 9 \sqrt{14}$$
$$A = 144 \sqrt{14}$$

I determined the square root of the first factor by dividing each exponent by 2, the index of the radical. Then I simplified the expression.

$$140 \cdot 4 = 560$$

I know $\sqrt{16} = 4$, so $\sqrt{14}$ must be a little less than 4. I rounded 144 to 140. The product must be less than 560.

The surface area of the mainsail is a little less than 560 ft^2.

Reflecting

A. Why does it make sense to use only the principal square root in this situation?

B. Why do you think Tim wrote the radicand as a product of prime factors?

C. Did Tim express this entire radical as a mixed radical in simplest form? Explain how you know.

APPLY the Math

EXAMPLE 2 Expressing a radical in simplest form

Express each entire radical in simplest form.

a) $\sqrt{3600}$ **b)** $\sqrt{288}$ **c)** $\sqrt[3]{432}$

Calvin's Solution

a) $\sqrt{3600} = \sqrt{2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 5}$

I factored 3600 into prime factors.

$= \sqrt{2^4 \cdot 3^2 \cdot 5^2}$

I wrote the factors as powers.

$= 2^2 \cdot 3 \cdot 5$

$= 4 \cdot 3 \cdot 5$

Each power had an even exponent. So, I divided each exponent by 2 to determine the square root of each power.

$= 60$

I multiplied the products to express $\sqrt{3600}$ in simplest form, which in this case is an integer with no radical.

b) $\sqrt{288} = \sqrt{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3}$

I factored 288 into prime factors.

$= \sqrt{2^5 \cdot 3^2}$

I wrote the factors as powers.

$= \sqrt{2^4 \cdot 2^1 \cdot 3^2}$

The index of this radical is 2. I wrote 2^5 as a product with an even exponent, so I could determine square roots.

$= \sqrt{2^4 \cdot 3^2 \cdot 2^1}$

I put the factors with even exponents together.

$= \sqrt{2^4 \cdot 3^2} \cdot \sqrt{2^1}$

I wrote the expression as the product of two radicals, one with the highest even exponents possible.

$= 2^2 \cdot 3^1 \cdot \sqrt{2}$

I wrote the radical with even exponents as the product of integers, by dividing the exponents by 2.

$= 4 \cdot 3 \cdot \sqrt{2}$

I expressed each power as an integer.

$= 12\sqrt{2}$

I multiplied the integers. I cannot simplify $\sqrt{2}$, since 2 is prime.

c) $\sqrt[3]{432} = \sqrt[3]{2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3}$ — I factored 432 into prime factors.

$= \sqrt[3]{2^4 \cdot 3^3}$ — I wrote the factors as powers.

$= \sqrt[3]{2^3 \cdot 2^1 \cdot 3^3}$ — The index of this radical is 3. I wrote 2^4 as a product with one exponent that is a multiple of 3, so I could determine cube roots.

$= \sqrt[3]{2^3 \cdot 3^3 \cdot 2^1}$ — I put the factors with exponents of 3 together.

$= \sqrt[3]{2^3 \cdot 3^3} \cdot \sqrt[3]{2^1}$ — I wrote the expression as the product of two radicals, one with exponents of 3.

$= 2 \cdot 3 \cdot \sqrt[3]{2}$ — I divided the exponents of the first radical by 3 to determine the cube root of each power.

$= 6\sqrt[3]{2}$ — I multiplied the integers. I cannot simplify $\sqrt{2}$, since 2 is prime.

Your Turn

Write $\sqrt{88\ 200}$ in simplest form.

EXAMPLE 3 Expressing a mixed radical as an entire radical

Express each mixed radical as an entire radical.

a) $4\sqrt{5}$ **b)** $-5\sqrt{2}$ **c)** $-2\sqrt[3]{5}$

Sebastian's Solution

a) $4\sqrt{5} = \sqrt{4^2} \cdot \sqrt{5}$ — I wrote 4 as the principal square root of 4^2.

$= \sqrt{4^2 \cdot 5}$ — I expressed the product of the two radicals as the radical of the product, which I could do since both radicals had the same index, 2.

$= \sqrt{16 \cdot 5}$ — I wrote 4^2 as a natural number.

$= \sqrt{80}$ — I multiplied to obtain the entire radical.

b) $-5\sqrt{2} = -1 \cdot 5\sqrt{2}$

The index of the radical is 2, so the radicand cannot be negative. I wrote the expression as the product of -1 and a mixed radical.

$= -1 \cdot \sqrt{25} \cdot \sqrt{2}$

I wrote 5 as the square root of 5^2, or $\sqrt{25}$.

$= -1 \cdot \sqrt{25 \cdot 2}$

I expressed the product of the two radicals as the radical of the product, which I could do since both radicals had the same index, 2.

$= -1\sqrt{50}$

$= -\sqrt{50}$

I multiplied the numbers under the radical. Since the original radical expression is a negative mixed radical, the entire radical must also be negative.

c) $-2\sqrt[3]{5} = \sqrt[3]{(-2)^3} \cdot \sqrt[3]{5}$

The index of the radical is 3, so the radicand can be negative. I wrote -2 as the cube root of $(-2)^3$.

$= \sqrt[3]{-8} \cdot \sqrt[3]{5}$

I wrote $(-2)^3$ as -8.

$= \sqrt[3]{-8 \cdot 5}$

I expressed the product of the two radicals as the radical of the product, which I could do since both radicals had the same index, 3.

$= \sqrt[3]{-40}$

I multiplied the values under the radical.

Your Turn

Express $7\sqrt{10}$ as an entire radical.

In Summary

Key Ideas

- You can express a radical in simplest form by using prime factors.
  For example:

$$\sqrt{75} = \sqrt{5^2 \cdot 3} = 5\sqrt{3} \qquad \sqrt[3]{375} = \sqrt[3]{5^3 \cdot 3} = 5\sqrt[3]{3}$$

  When expressing square roots in simplest form, try to combine prime factors to create powers with even exponents. When working with cube roots, try to create powers with exponents that are multiples of 3.
- You can express a mixed radical as an entire radical, by writing the leading number as a radical, then multiplying the radicands.

  For example:

$$4\sqrt{2} = \sqrt{16} \cdot \sqrt{2} = \sqrt{16 \cdot 2} = \sqrt{32}$$

Need to Know

- A radical is in simplest form when the exponent of the radicand is less than the index of the radical. For example, $12\sqrt{3}$ and $13\sqrt[3]{4}$ are in simplest form, while $12\sqrt{4}$ is not.
- A square has a principal square root, which is positive, and a secondary square root, which is negative. For example, the principal square root of 16 is $\sqrt{16}$ or 4, and the secondary square root of 16 is $-\sqrt{16}$ or -4. The radical of a square root may be negative, but the radicand of a square root must be positive.
- If you express an answer as a radical, the answer will be exact. If you write a radical in decimal form, the answer will be an approximation, except when the radicand is a perfect square. For example, $\sqrt{12}$ expressed as $2\sqrt{3}$ remains an exact value, while $\sqrt{12}$ expressed as 3.464… is an approximation. Both $\sqrt{9}$ and 3 are exact values.

CHECK Your Understanding

1. True or false? Explain why.

a) The principal square root of 25 is -5.

b) $\sqrt{16} = \pm 4$

c) $\sqrt[3]{(-4)^3} = \pm 4$

d) $\sqrt{-4} = -2$

e) $-\sqrt{4} = -2$

f) $\sqrt{36} = -6$

2. Match each radical on the left with its equivalent value on the right.

a) $2\sqrt{31}$ **i)** $4\sqrt{7}$

b) $\sqrt{112}$ **ii)** $2\sqrt[3]{15}$

c) $8\sqrt{3}$ **iii)** $2\sqrt{48}$

d) $\sqrt[3]{120}$ **iv)** 11.1

3. Given: $\sqrt{432}$ and $5\sqrt[3]{2}$

a) Which is expressed as an entire radical and which is a mixed radical?

b) Write each radical in its alternative form.

PRACTISING

4. Express each radical as a mixed radical in simplest form.

a) $\sqrt{72}$ **b)** $\sqrt{600}$ **c)** $\sqrt[3]{40}$ **d)** $\sqrt[3]{250}$

5. Express each radical as a mixed radical in simplest form.

a) $\sqrt{4000}$ **b)** $-\sqrt{2835}$ **c)** $\sqrt[3]{-72}$ **d)** $\sqrt[3]{648}$

6. Express each radicand as a product of primes with the greatest possible exponents.

a) $\sqrt{3888}$ **b)** $\sqrt{100\,000}$ **c)** $\sqrt{16\,000}$ **d)** $\sqrt{16\,875}$

7. Express each radical as a product of two radicals.

a) $\sqrt{196}$ **b)** $\sqrt{3600}$ **c)** $\sqrt[3]{64}$ **d)** $\sqrt[3]{3375}$

8. Express each radical as an integer using two different groups of exponents.

a) $\sqrt{64}$ **b)** $\sqrt{3600}$ **c)** $\sqrt[3]{8000}$

9. Kenny states that he has demonstrated that $-\sqrt{16} = 4$. What error did Kenny make?

$$\begin{aligned} (-4) &= -4 \\ \sqrt{(-4)(-4)} &= -4 \\ \sqrt{16} &= -4 \\ -\sqrt{16} &= 4 \end{aligned}$$

10. Estimate the value of each radical. Then evaluate it to the nearest hundredth.

a) $\sqrt{47}$ **b)** $\sqrt{41\,000}$ **c)** $\sqrt{790}$ **d)** $\sqrt[3]{900}$

11. Express each mixed radical as an entire radical.

a) $6\sqrt{5}$ b) $12\sqrt{7}$ c) $4\sqrt[3]{14}$ d) $-3\sqrt[3]{4}$

12. Given: 4, $4\sqrt{3}$, $\sqrt{14}$, $3\sqrt{2}$, $4\sqrt{5}$

a) Write each number as an entire radical.

b) Arrange the numbers in increasing order.

13. Manuel insists that he can add $2\sqrt{4}$ and $5\sqrt[3]{4}$ in radical form because both values are positive. Do you agree or disagree? Justify your answer.

14. The number 1 is a natural number whose principal square root and cube root are also natural numbers: $\sqrt{1} = 1$ and $\sqrt[3]{1} = 1$. Determine two other natural numbers with this property.

15. Write all the mixed radicals that are equivalent to $\sqrt{800}$. Which is written in simplest form? Explain how you know.

16. The speed of an airplane is given by $S = 0.1\sqrt{L}$, where S represents the speed in metres per second and L represents the lift, in newtons (N).

a) Suppose the lift for a particular airplane is 810 000 N. Determine the speed of the plane to the nearest metre per second.

b) Express your answer for part a) in kilometres per hour. Is this a reasonable speed for a plane? Explain.

17. Highway engineers design on-ramps and off-ramps to be safe and efficient. The relationship between the maximum speed at which a car can safely travel around a flat curve without skidding is

$$S = \sqrt{6.86\,R}$$

where S represents the maximum speed, in metres per second, and R represents the radius of the curve, in metres. What is the maximum speed, in kilometres per hour, at which a car can safely travel on a ramp with a radius of 50 m?

Closing

18. A school playground covers an area of $\sqrt{14\,000}$ m^2.

a) Express this radical as a mixed radical in simplest form. For your first step, express the radicand as the product of prime factors.

b) Could this playground be square? Explain.

Extending

19. A Pythagorean triple is a set of three natural numbers that satisfy the Pythagorean theorem; for example, 3, 4, and 5 since $3^2 + 4^2 = 5^2$. Another way to write this would be $9 + 16 = 25$. List two other sets of Pythagorean triples. Explain how you used radicals to locate these numbers.

20. Is each statement true or false? Justify your answer.

a) $(\sqrt{x})^4 = x^2$ b) $(\sqrt{x})^3 = x\sqrt{x}$

4.2 Adding and Subtracting Radicals

YOU WILL NEED

- square piece of paper

GOAL

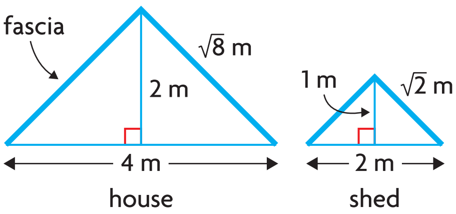
Select a strategy to add two radicals.

EXPLORE...

- Fold a square piece of paper in half to produce a rectangle. Tear apart the two halves. Fold one rectangle in half to produce a square. Fold the square along its diagonal to produce a triangle. Fold the triangle in half. Unfold your paper and determine the dimensions of the smallest triangle.

LEARN ABOUT the Math

Karen's uncle is planning to replace the fascia on one side of the roof of his house and gardening shed. Fascia is the flat surface immediately below the edge of a roof. He has determined the measurements of the sides of the roofs using the Pythagorean theorem.



? What length of fascia is needed?

EXAMPLE 1 Adding radicals

Determine the length of fascia needed.

Quinton's Solution: Adding radicals by reducing them to simplest form

The length of fascia needed, L, is:

$$L = \sqrt{8} + \sqrt{8} + \sqrt{2} + \sqrt{2}$$

The sides of the house roof are each $\sqrt{8}$ m and those of the shed are each $\sqrt{2}$ m, so I needed to add the lengths. The $\sqrt{8}$ and $\sqrt{2}$ are not like radicals, so I could not add them at this point.

$$L = \sqrt{4} \cdot \sqrt{2} + \sqrt{4} \cdot \sqrt{2} + \sqrt{2} + \sqrt{2}$$

$$L = 2\sqrt{2} + 2\sqrt{2} + 1\sqrt{2} + 1\sqrt{2}$$

I wrote the $\sqrt{8}$ radicals in simplest form. $\sqrt{2}$ is in simplest form already.

$L = 6\sqrt{2}$

I know that $2x + 2x + x + x = 6x$ when I combine like terms. So, I can add the $\sqrt{2}$ radical terms in the same way to get my answer of $6\sqrt{2}$.

$L \doteq 6 \cdot 1.5$
$L \doteq 9$ m

I need to estimate because the exact answer won't help Karen's uncle at the hardware store. $\sqrt{2}$ is a little less than 1.5.

Karen's uncle should buy 9 m of fascia.

This will be a little more than he needs.

Tyron's Solution: Adding radicals using a calculator

The length of fascia needed, L, is:

$L = \sqrt{8} + \sqrt{8} + \sqrt{2} + \sqrt{2}$

$L \doteq 8.485\ldots$

I entered each term in the radical expression on my calculator and added.

Karen's uncle needs about 8.5 m of fascia.

I rounded to the nearest tenth to ensure he has enough material.

Reflecting

A. How would calculating $\sqrt{8} - \sqrt{2}$ be the same as adding $\sqrt{8} + \sqrt{2}$? How would it be different?

B. Todd claims that Tyron could have arrived at his answer two different ways. Do you agree or disagree? Explain.

C. In the third line of Quinton's Solution, why do you think he wrote each $\sqrt{2}$ as $1\sqrt{2}$?

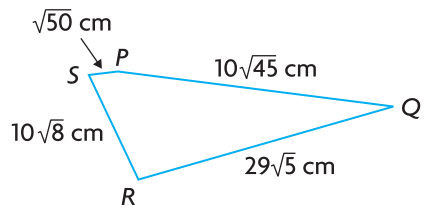
APPLY the Math

EXAMPLE 2 Subtracting radicals

Determine the difference in length between each pair of sides.

a) PS and SR

b) RQ and PQ



Paula's Solution

a) Let D represent the difference between PS and SR.

$D = PS - SR$

$D = \sqrt{50} - 10\sqrt{8}$

$D = 5\sqrt{2} - 20\sqrt{2}$

I expressed the side lengths in simplest form.

$D = -15\sqrt{2}$

I subtracted the like radicals.

The difference between PS and SR is $15\sqrt{2}$ cm.

My answer is negative, because I subtracted a greater number from a smaller one, however, I know length can only be positive.

b) Let E represent the difference between RQ and PQ.

$E = RQ - PQ$

$E = 29\sqrt{5} - 10\sqrt{45}$

$E = 29\sqrt{5} - 30\sqrt{5}$

I expressed the side lengths in simplest form.

$E = -\sqrt{5}$

I subtracted the like radicals. Again, I know length can't be negative, so my answer is positive.

The difference between RQ and PQ is $\sqrt{5}$ cm.

Your Turn

Determine the difference between a side of length $\sqrt{243}$ cm and one of $15\sqrt{3}$ cm.

EXAMPLE 3 Simplifying radical expressions

Simplify each expression.

a) $2\sqrt{27} - 4\sqrt{3} - \sqrt{12}$ **b)** $2\sqrt{24} - 3\sqrt{96} + \sqrt{432}$

Vanessa's Solution

a) $2\sqrt{27} - 4\sqrt{3} - \sqrt{12}$

$= 2\sqrt{9} \cdot \sqrt{3} - 4\sqrt{3} - \sqrt{4} \cdot \sqrt{3}$

I determined the simplest form of each term.

$= 2 \cdot 3\sqrt{3} - 4\sqrt{3} - 2\sqrt{3}$

$= 6\sqrt{3} - 4\sqrt{3} - 2\sqrt{3}$

The terms were all like radicals, so I was able to subtract.

$= 0\sqrt{3}$

$= 0$

I subtracted from left to right, as with any calculation.

b) $2\sqrt{24} - 3\sqrt{96} + \sqrt{432}$

$= (2\sqrt{2^2} \cdot \sqrt{2 \cdot 3}) - (3\sqrt{2^4} \cdot \sqrt{2 \cdot 3}) + (\sqrt{2^4 \cdot 3^2} \cdot \sqrt{3})$

I determined the simplest form of each term in the expression.

$= 2 \cdot 2\sqrt{6} - 3 \cdot 2^2\sqrt{6} + 2^2 \cdot 3\sqrt{3}$

$= 4\sqrt{6} - 12\sqrt{6} + 12\sqrt{3}$

I subtracted the like radicals, but I couldn't add the unlike radical.

$= -8\sqrt{6} + 12\sqrt{3}$

Your Turn

Create a negative mixed radical using one addition sign, one subtraction sign, and the radicals $\sqrt{32}$, $3\sqrt{8}$, and $\sqrt{18}$.

In Summary

Key Ideas

- Radicals with the same radicand and index are considered like radicals.
- You can add or subtract like radicals. For example:

  $6\sqrt{3} + 2\sqrt{3} = (6 + 2)\sqrt{3}$, or $8\sqrt{3}$

  $6\sqrt{3} - 2\sqrt{3} = (6 - 2)\sqrt{3}$, or $4\sqrt{3}$
- You cannot add or subtract unlike radicals, such as $6\sqrt{2}$ and $4\sqrt{5}$.

Need to Know

- When you add or subtract radicals, it helps to express each one in its simplest form first.
- When calculating, keep numbers in radical form until the end of the solution, so you can determine the exact value instead of an approximation.

CHECK Your Understanding

1. Are the radicals in each set like or unlike radicals?
   a) $4\sqrt{2}, 5\sqrt{2}, 12\sqrt{2}, -3\sqrt{2}$
   b) $4\sqrt[3]{2}, 7\sqrt[3]{3}, 6\sqrt{2}, -6\sqrt{2}$
   c) $6\sqrt{5}, -5\sqrt{5}, 4\sqrt{5}, 4\sqrt{125}$
   d) $\sqrt[3]{-8}, -\sqrt{16}, 2\sqrt{4}, 4\sqrt{4}$

2. Simplify.
   a) $4\sqrt{6} + 3\sqrt{6} + 2\sqrt{6}$
   b) $15\sqrt{3} - 3\sqrt{3} - 8\sqrt{3}$
   c) $5\sqrt{2} - 9\sqrt{2} - \sqrt{2} + 11\sqrt{2}$
   d) $-7\sqrt{10} - 4\sqrt{10} - 3\sqrt{10} + 12\sqrt{10}$

3. Simplify $\sqrt{12} + 2\sqrt{27}$. Explain each step.

4. Write in mixed radical form, then simplify.
   a) $\sqrt{8} - \sqrt{32} + \sqrt{512}$
   b) $-\sqrt{27} + \sqrt{75} - \sqrt{12}$

PRACTISING

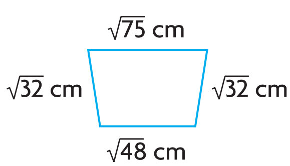
5. Simplify.
   a) $\sqrt{72} + \sqrt{50}$
   b) $7\sqrt{3} + 2\sqrt{45} + \sqrt{108}$
   c) $\sqrt{32} + 5\sqrt{2} + \sqrt{400}$
   d) $3\sqrt{20} + 4\sqrt{60} + \sqrt{125}$

6. Simplify.
   a) $\sqrt{40} - \sqrt{360}$
   b) $6\sqrt{27} - \sqrt{75} - 4\sqrt{48}$
   c) $5\sqrt{32} - 7\sqrt{2} - \sqrt{484}$
   d) $3\sqrt{18} - 6\sqrt{45} - 5\sqrt{108}$

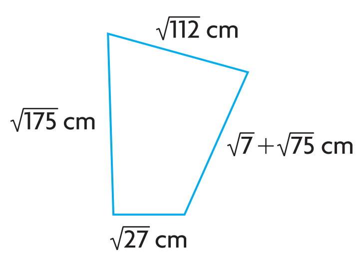
7. Kimmi wants to sew a ribbon border around a small triangular cushion with sides of $\sqrt{63}$ cm, $\sqrt{50}$ cm, and $\sqrt{72}$ cm.
   a) Determine the exact length of ribbon Kimmi needs.
   b) Determine the length Kimmi needs to a tenth of a centimetre.

8. The sum of any two sides of a triangle must be greater than the third side. Can you create a triangle with side lengths of $\sqrt{28}$ cm, $\sqrt{63}$ cm, and $\sqrt{147}$ cm? Explain your answer.

9. Express the perimeter of each figure in simplest form.
   a)

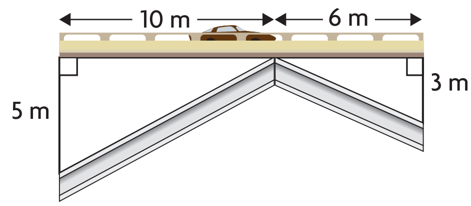


   b)



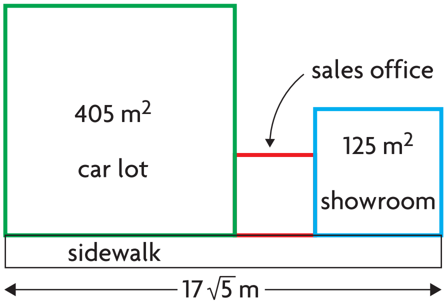
10. John has two small square cushions, one 400 cm^2 in area and the other 578 cm^2 in area. Determine, in two different ways, the combined perimeter of the two cushions. Provide an exact answer.

11. A design for an overpass is shown. Determine the total length of steel needed to form the angled support section of the bridge. Express your answer as a radical in simplest form.



12. This house covers a square with an area of 90 m^2, and the garage covers a square with an area of 40 m^2. Determine the combined length of the front of the house and garage as a radical in simplest form.

13. An architect is designing the floor plan of a car dealership, as shown, with the sales office between the square showroom and the square car lot. The front of the entire dealership is exactly $17\sqrt{5}$ m long. Determine the exact width of the sales office.



14. Express $3\sqrt{32} - \sqrt{8} + 2\sqrt{50} - \sqrt{18}$ in simplest form.

15. Express $3\sqrt{80} - 5\sqrt{150} + 4\sqrt{384} - 3\sqrt{45}$ in simplest form.

16. James was asked to express $12\sqrt{24} + 12\sqrt{3}$ in simplest form. He wrote the following:

$$\begin{aligned} 12\sqrt{24} + 12\sqrt{3} &= 12\sqrt{2 \cdot 2 \cdot 2 \cdot 3} + 12\sqrt{3} \\ &= 12 \cdot \sqrt{2^2} \cdot \sqrt{2 \cdot 3} + 12\sqrt{3} \\ &= 12 \cdot 2 \cdot \sqrt{6} + 12\sqrt{3} \\ &= 24\sqrt{6} + 12\sqrt{3} \\ &= 36\sqrt{9} \end{aligned}$$

What error did James make? What should the answer be?

17. Explain how to determine how many different terms a radical expression will have in its simplest form. Support your explanation with an example.

18. Express $\sqrt{600} - \sqrt{486} + \sqrt{150}$ in simplest form.

19. Write each expression in simplest form.

a) $\sqrt{50} - 6\sqrt{2}$ **b)** $-3\sqrt{3} + \sqrt{192}$ **c)** $\sqrt{3125} - 2\sqrt{5} - 3\sqrt{20}$

Closing

20. How is adding and subtracting radicals, such as $3\sqrt{42} + 4\sqrt{36}$, the same as adding and subtracting algebraic terms, such as $3x + 4x$? How is it different? Explain.

Extending

21. Jasmine drew two squares on graph paper. Square A had an area of 2 square units and Square B had an area of 8 square units. How many times longer are the sides of Square B than the sides of Square A? Determine your answer using your knowledge of adding radicals. Confirm your answer using graph paper.

History Connection

It's Radical

Radical numbers, such as $\sqrt{2}$, have been known since at least the time of Pythagoras, more than 2000 years ago. But it was "only" about 800 years ago, in the 1200s, that mathematicians started to use symbols to indicate radicals. Originally, mathematicians used the Latin word *radix*, which means "root," to indicate that a number was a radical. Then they shortened that, just using the letter "R" with a line crossing its right leg, much like the symbol used in today's medical prescriptions.

The radical sign as we know it today first appeared in print in the 1500s, in a German book on algebra by Christoff Rudolff. (This book also introduced the signs "+" and "−" to the world.)

A. Suppose Jackie is asked to solve this problem:

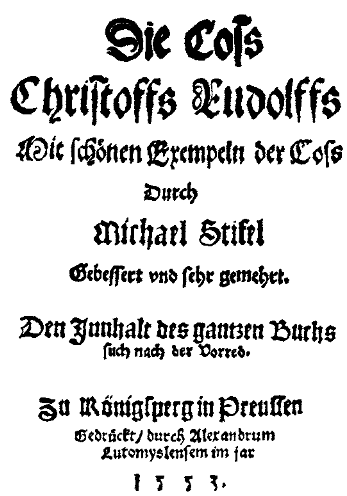
$$\sqrt{4x + 12x} - \sqrt{2x + 2x} = 4$$

Suppose Jim is asked to solve this problem:

The difference between the radix of the sum of four equal measures and twelve equal measures and the radix of the sum of two equal measures and two equal measures is equal to four units. Determine the number of units in one equal measure.

Who do you think would solve their problem more quickly? Explain.

B. How did other mathematical symbols develop over the centuries? Pick two and present the results of your research.


Die Coſs
Chriſtoffs Rudolffs
Die ſchönen Exempeln der Coſs
Durch
Michael Stifel
Gebeſſert vnd ſehr gemehret.
Den Innhalt des gantzen Buchs
ſuch nach der Vorred.
Zu Königſperg in Preuſſen
Gedrückt / durch Alexandrum
Lutomyslenſem im jar
1553.

The title page of *Die Coss,* Christoff Rudolff's algebra text

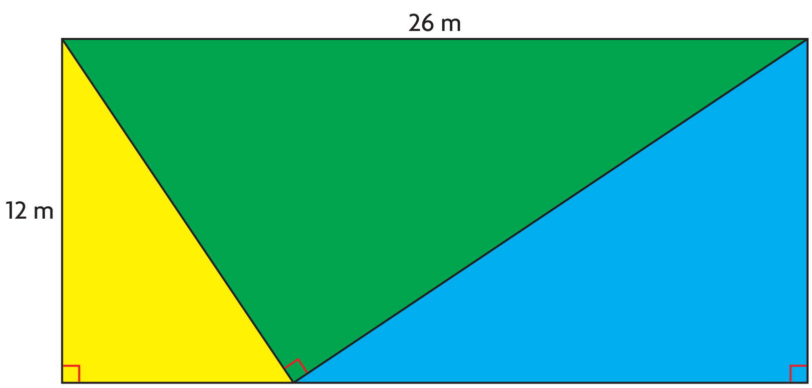
4.3 Multiplying and Dividing Radicals

GOAL

Multiply and divide numerical radicals.

INVESTIGATE the Math

For a school logo, Hugo will paint the north wall of the gym with yellow, green, and blue right triangles, as shown. The height of the yellow triangle and the length of its base will be in a 3 : 2 ratio. A 4 L can of paint covers about 32.5 m^2. Hugo plans to apply two coats of paint.



? How much green paint should Hugo buy?

A. Determine the dimensions of each triangle in radical form.

B. Express the area of the green triangle in simplest form.

C. Find a classmate who used the formula for area of a triangle using a different method from yours. How are your solutions alike? How are they different? Explain.

D. How many cans of green paint does Hugo need to buy?

Reflecting

E. Why was it useful to keep the dimensions of the green triangle in the form of radicals?

F. How can you verify your answer without using radicals?

EXPLORE...

- A bowl contains 36 cubes of sugar. Each sugar cube is 1 cm on a side. Cathy stacks the cubes in a rectangular prism with the smallest possible surface area. What is the length of the longest diagonal measure in her prism?

APPLY the Math

EXAMPLE 1 Multiplying radicals with the same index

Show that the following equations are true.

a) $\sqrt{3} \cdot \sqrt{5} = \sqrt{15}$ b) $\dfrac{\sqrt{3}}{\sqrt{5}} = \sqrt{\dfrac{3}{5}}$

Marcus's Solution

LS	RS
a) $\sqrt{3} \cdot \sqrt{5}$	$\sqrt{15}$
$3^{\frac{1}{2}} \cdot 5^{\frac{1}{2}}$	
$(3 \cdot 5)^{\frac{1}{2}}$	
$15^{\frac{1}{2}}$	
$\sqrt{15}$	

I know that the square root of a number is the same as the radicand having an exponent of $\frac{1}{2}$.

By the laws of exponents, I can apply the same exponent to the product.

I wrote $15^{\frac{1}{2}}$ as a square root. This proves that the expressions are equal.

$$\sqrt{3} \cdot \sqrt{5} = \sqrt{15}$$

LS	RS
b) $\dfrac{\sqrt{3}}{\sqrt{5}}$	$\sqrt{\dfrac{3}{5}}$
$\dfrac{3^{\frac{1}{2}}}{5^{\frac{1}{2}}}$	
$\left(\dfrac{3}{5}\right)^{\frac{1}{2}}$	
$\dfrac{\sqrt{3}}{\sqrt{5}}$	
$\sqrt{\dfrac{3}{5}}$	

I can use a similar proof to that in part a).

By the laws of exponents, I can apply the same exponent to the quotient.

I wrote $\left(\frac{3}{5}\right)^{\frac{1}{2}}$ as a square root.

This proves that the expressions are equal.

Your Turn

Bryce makes this claim:

$$\sqrt{9} \cdot \sqrt{8} = 6\sqrt{2}$$

Is he correct? Explain.

EXAMPLE 2 Multiplying radicals using the distributive property

Express in simplest form.

a) $4\sqrt{2}(7\sqrt{5} + \sqrt{3})$ b) $(5\sqrt{3} + 2\sqrt{6})^2$

Luba's Solution

a) $4\sqrt{2}(7\sqrt{5} + \sqrt{3})$

$= 4\sqrt{2} \cdot 7\sqrt{5} + 4\sqrt{2} \cdot \sqrt{3}$

I cannot add the radicals in the brackets because they are not alike. I expanded by distributing $4\sqrt{2}$ to each term in the brackets.

$= 28\sqrt{10} + 4\sqrt{6}$

I multiplied the integers and the radicals of each term. I could not reduce the radicals any further, so this is the expression in simplest form.

b) $(5\sqrt{3} + 2\sqrt{6})^2$

$= (5\sqrt{3} + 2\sqrt{6}) \cdot (5\sqrt{3} + 2\sqrt{6})$

I wrote the expression as a product.

$= 5\sqrt{3} \cdot 5\sqrt{3} + 5\sqrt{3} \cdot 2\sqrt{6} + 2\sqrt{6} \cdot 5\sqrt{3} + 2\sqrt{6} \cdot 2\sqrt{6}$

I expanded the binomial.

$= 25\sqrt{9} + 10\sqrt{18} + 10\sqrt{18} + 4\sqrt{36}$

$= 25 \cdot 3 + 20\sqrt{18} + 4 \cdot 6$

I evaluated the perfect squares and added the like radicals.

$= 75 + 24 + 20\sqrt{18}$

$= 99 + 20\sqrt{18}$

$= 99 + 20\sqrt{9} \cdot \sqrt{2}$

$= 99 + 20 \cdot 3\sqrt{2}$

$= 99 + 60\sqrt{2}$

I simplified.

Your Turn

Express $2\sqrt{3}(\sqrt{12} - \sqrt{7})$ in simplest form.

EXAMPLE 3 Dividing radicals by a monomial

Express each of the following in simplest form.

a) $\dfrac{6\sqrt{48}}{3\sqrt{6}}$ **b)** $\dfrac{4\sqrt{12} - 10\sqrt{6}}{2\sqrt{3}}$

Benito's Solution: Calculating rational numbers and radicals separately

a) $\dfrac{6\sqrt{48}}{3\sqrt{6}}$

$= \dfrac{6}{3} \cdot \sqrt{\dfrac{48}{6}}$

The mixed radicals in the numerator and the denominator are products.

I wrote the expression as a product of an integer quotient and a single radical quotient.

$= 2 \cdot \sqrt{8}$

I simplified by dividing.

$= 2 \cdot \sqrt{2^2 \cdot 2}$
$= 2 \cdot 2\sqrt{2}$
$= 4\sqrt{2}$

I expressed $\sqrt{8}$ as a mixed radical using prime factorization and then multiplied the two rational numbers.

b) $\dfrac{4\sqrt{12}-10\sqrt{6}}{2\sqrt{3}}$

$= \dfrac{4\sqrt{12}}{2\sqrt{3}} - \dfrac{10\sqrt{6}}{2\sqrt{3}}$

I wrote the expression as a subtraction statement involving two terms so I could work with each term separately.

$= \left(\dfrac{4}{2} \cdot \sqrt{\dfrac{12}{3}}\right) - \left(\dfrac{10}{2} \cdot \sqrt{\dfrac{6}{3}}\right)$

I wrote each term as the product of an integer quotient and a single radical quotient.

$= 2 \cdot \sqrt{4} - 5 \cdot \sqrt{2}$

I simplified the terms by dividing.

$= 2 \cdot 2 - 5\sqrt{2}$

I wrote $\sqrt{4}$ as 2. I couldn't reduce $\sqrt{2}$ any further, since 2 is a prime number.

$= 4 - 5\sqrt{2}$

I multiplied the two rational numbers to obtain the simplest form.

Yvette's Solution: Rationalizing the denominator

a) $\dfrac{6\sqrt{48}}{3\sqrt{6}}$

$= \dfrac{2\sqrt{48}}{\sqrt{6}}$

I divided the numerator and denominator by the common factor 3 to eliminate the coefficient in the denominator.

$= \dfrac{2\sqrt{48}}{\sqrt{6}} \cdot \dfrac{\sqrt{6}}{\sqrt{6}}$

$= \dfrac{2\sqrt{48} \cdot \sqrt{6}}{6}$

I wanted to **rationalize the denominator,** so I multiplied the expression by $\frac{\sqrt{6}}{\sqrt{6}}$. Since this is equal to 1, the value of the expression did not change.

$= \dfrac{2\sqrt{8} \cdot \sqrt{6} \cdot \sqrt{6}}{6}$

$= \dfrac{2\sqrt{8} \cdot 6}{6}$

$= 2\sqrt{8}$

I expressed $\sqrt{48}$ as a product of two radicals, $\sqrt{8}$ and $\sqrt{6}$, because $\sqrt{6^2}$ eliminates the radical sign. I simplified using the fact that $\frac{6}{6} = 1$.

$= 2 \cdot 2\sqrt{2}$

I wrote $\sqrt{8}$ in simplest form.

$= 4\sqrt{2}$

I multiplied to obtain the simplest form.

b) $\dfrac{4\sqrt{12} - 10\sqrt{6}}{2\sqrt{3}}$

$= \dfrac{2\sqrt{12} - 5\sqrt{6}}{\sqrt{3}}$

I divided each term in the numerator and denominator by the common factor 2 to eliminate the integer in the denominator.

$= \dfrac{(2\sqrt{12} - 5\sqrt{6})}{\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}}$

I multiplied the expression by 1, in the form $\frac{\sqrt{3}}{\sqrt{3}}$, so I could express the radical in the denominator as a rational number.

$= \dfrac{2\sqrt{36} - 5\sqrt{18}}{3}$

I multiplied both terms in the numerator and the term in the denominator by $\sqrt{3}$.

Communication *Tip*

By convention, a radical expression is in simplest form only when the denominator of the expression is a rational number.

rationalize the denominator

The process used to write a radical expression that contains a radical denominator as an equivalent expression with a rational denominator.

$$= \frac{2 \cdot 6 - 5 \cdot 3\sqrt{2}}{3}$$

I simplified the radicals.

$$= \frac{12 - 15\sqrt{2}}{3}$$

I multiplied to produce a new numerator.

$$= 4 - 5\sqrt{2}$$

I divided each term by the common factor of 3 to eliminate the denominator and obtain the simplest form.

Maria's Solution: Eliminating common factors

a) $\frac{6\sqrt{48}}{3\sqrt{6}}$

$$= \frac{2 \cdot 3 \cdot \sqrt{6} \cdot \sqrt{8}}{3 \cdot \sqrt{6}}$$

I factored the numerator. I noticed that it and the denominator had the common factor of $3\sqrt{6}$.

$$= 2\sqrt{8}$$

I eliminated the denominator by dividing by $3\sqrt{6}$.

$$= 2 \cdot \sqrt{2^3}$$

I wrote $\sqrt{8}$ as a mixed radical.

$$= 2 \cdot \sqrt{2^2} \cdot \sqrt{2}$$

$$= 2 \cdot 2\sqrt{2}$$

$$= 4\sqrt{2}$$

I multiplied the rational numbers to write the expression in simplest form.

b) $\frac{4\sqrt{12} - 10\sqrt{6}}{2\sqrt{3}}$

$$= \frac{2\sqrt{3} \cdot (2\sqrt{4} - 5\sqrt{2})}{2\sqrt{3}}$$

I factored the numerator. I noticed that it and the denominator had the common factor of $2\sqrt{3}$.

$$= 2\sqrt{4} - 5\sqrt{2}$$

I eliminated the denominator by dividing by $2\sqrt{3}$.

$$= 2 \cdot 2 - 5\sqrt{2}$$

I wrote $\sqrt{4}$ as 2.

$$= 4 - 5\sqrt{2}$$

I multiplied the rational numbers to obtain the simplest form.

Your Turn

Compare Yvette's Solution, Benito's Solution, and Maria's Solution. What are the advantages and disadvantages of each strategy?

In Summary

Key Idea

- You can use the same properties you use with rational numbers to multiply and divide radical numbers:
  - the commutative property; for example:
    $$5\sqrt{2} = \sqrt{2} \cdot 5$$
  - the associative property; for example:
    $$\sqrt{3}(5\sqrt{2}) = 5(\sqrt{3} \cdot \sqrt{2}) \text{ or } \sqrt{2}\,(5\sqrt{3})$$
  - the distributive property; for example:
    $$\sqrt{4}(\sqrt{2} + 1) = \sqrt{4} \cdot \sqrt{2} + \sqrt{4} \cdot 1$$
  - the multiplicative identity property; for example:
    $$\sqrt{5} \cdot 1 = \sqrt{5} \cdot \frac{\sqrt{5}}{\sqrt{5}}$$

Need to Know

- The product of two square roots is equal to the square root of the product.

  $\sqrt{a} \cdot \sqrt{b} = \sqrt{a \cdot b}$ For example: $\sqrt{3} \cdot \sqrt{2} = \sqrt{3 \cdot 2}$ or $\sqrt{6}$

  when $a \geq 0$, $b \geq 0$
- The product of two mixed radicals is equal to the product of the rational numbers times the product of the radicals.

  $c\sqrt{a} \cdot d\sqrt{b} = c \cdot d\sqrt{ab}$ For example: $3\sqrt{2} \cdot 5\sqrt{7} = 3 \cdot 5 \cdot \sqrt{2} \cdot \sqrt{7}$ or $15\sqrt{14}$

  when $a \geq 0$, $b \geq 0$
- The quotient of two square roots is equal to the square root of the quotient.

  $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$ For example: $\dfrac{\sqrt{6}}{\sqrt{2}} = \sqrt{\dfrac{6}{2}}$ or $\sqrt{3}$

  when $a \geq 0$, $b > 0$
- The quotient of two mixed radicals is equal to the product of the quotient of the coefficients and the quotient of the radicals.

  $\dfrac{c\sqrt{a}}{d\sqrt{b}} = \dfrac{c}{d}\sqrt{\dfrac{a}{b}}$ For example: $\dfrac{15\sqrt{14}}{5\sqrt{7}} = \dfrac{15}{5} \cdot \sqrt{\dfrac{14}{7}}$ or $3\sqrt{2}$

  when $a \geq 0$, $b > 0$, $d \neq 0$
- One way to simplify an expression with a radical in the denominator is called rationalizing the denominator. To do this, multiply by 1 in a form that will change the denominator to a rational number.

  For example: $\dfrac{3\sqrt{7}}{4\sqrt{5}} = \dfrac{3\sqrt{7}}{4\sqrt{5}} \cdot \dfrac{\sqrt{5}}{\sqrt{5}}$ or $\dfrac{3\sqrt{35}}{20}$

CHECK Your Understanding

1. Write each expression in simplest form.

a) $\sqrt{5} \cdot \sqrt{6}$

b) $\sqrt{12} \cdot \sqrt{20}$

c) $2\sqrt{3} \cdot \sqrt{24}$

d) $7\sqrt{32} \cdot 2\sqrt{48}$

2. a) Write $\dfrac{1}{\sqrt{5}}$ in its rationalized form.

b) Explain why you can multiply $\dfrac{1}{\sqrt{5}}$ by $\dfrac{\sqrt{5}}{\sqrt{5}}$ without changing its value.

3. Simplify $\dfrac{\sqrt{64}}{\sqrt{4}}$ using three different methods. Explain what you did in each method.

PRACTISING

4. Express each product in mixed radical form and entire form.

a) $\sqrt{12} \cdot \sqrt{24}$

b) $3\sqrt{15} \cdot 2\sqrt{10}$

c) $-1\sqrt{30} \cdot \sqrt{54}$

d) $-2\sqrt{14} \cdot -1\sqrt{21}$

5. Expand each expression and simplify.

a) $7(3 + \sqrt{12})$

b) $\sqrt{5}(4 - \sqrt{10})$

c) $\sqrt{6}(\sqrt{10} - 8\sqrt{3})$

d) $2\sqrt{3}(\sqrt{18} + 5\sqrt{2})$

e) $(6 + \sqrt{6})(5 + \sqrt{10})$

f) $(2\sqrt{3} - 5\sqrt{8})^2$

6. a) Simplify $2\sqrt{12} \cdot \sqrt{18} \cdot 5\sqrt{6}$ in two different ways.

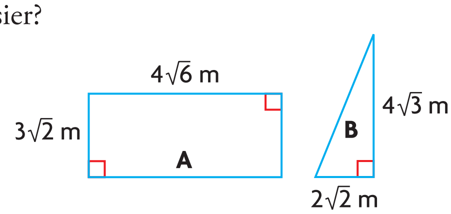
b) Which property (associative, commutative, or distributive) allowed you to do this? Explain.

7. Given: $\sqrt{192}$ and $\sqrt{4800}$

a) Express each radical as a product of smaller radicals, each with a different radicand.

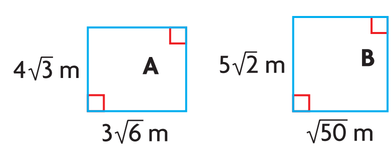
b) Write each radical in simplest form. How did rewriting each radical first make this step easier?

8. Determine which figure covers the greater area. Do not use a calculator.



9. a) Determine which figure covers the greater area.

b) Did you need to express the areas in decimal form to answer part a)? Explain.



10. Clare was asked to write $\sqrt{8}$ in simplest form. She wrote:

$$\begin{aligned}\sqrt{8} &= \sqrt{2 + 2 + 2 + 2}\\ &= \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2}\\ &= 4\sqrt{2}\end{aligned}$$

Identify the error in Clare's reasoning and write the correct answer.

11. Steve and Danny were asked to evaluate $(\sqrt{2} - 1)^2$.
  a) Who is incorrect? Justify your answer.
  b) Complete the correct solution.

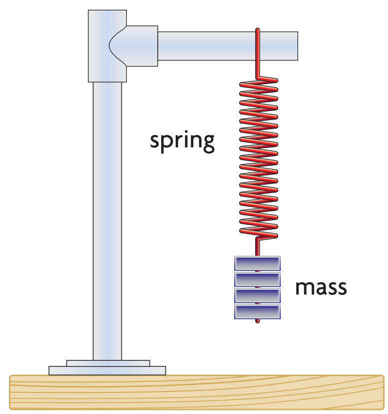
Steve's Solution	**Danny's Solution**
$(\sqrt{2} - 1)^2$	$(\sqrt{2} - 1)^2$
$(\sqrt{2})^2 - 1^2$	$(\sqrt{2} - 1)(\sqrt{2} - 1)$

12. The time it takes for a spring with a mass attached to move up and down until reaching its original position (one complete cycle) can be approximated by

$$T = 2\pi\sqrt{\frac{M}{K}}$$

where T represents the time in seconds, M represents the object's mass in kilograms, and K represents the spring's constant in newtons per metre (N/m).

Consider a spring holding a mass of 4 kg with a spring constant of 2 N/m.
  a) Determine T in simplest form.
  b) Determine T to the nearest hundredth of a second.



13. Rationalize the denominator in each expression.
  a) $\dfrac{\sqrt{7}}{\sqrt{2}}$ **b)** $\dfrac{-1}{4\sqrt{5}}$ **c)** $\dfrac{-3\sqrt{8}}{\sqrt{6}}$ **d)** $\dfrac{\sqrt{72}}{2\sqrt{8}}$

14. Write each expression in simplest form.
  a) $\dfrac{\sqrt{12}}{\sqrt{3}}$ **b)** $\dfrac{4\sqrt{15}}{-1\sqrt{5}}$ **c)** $\dfrac{-3\sqrt{30}}{\sqrt{6}}$ **d)** $\dfrac{-2\sqrt{98}}{\sqrt{8}}$

15. Sasha claims that she can simplify $\sqrt{0.16}$ by first representing the radicand as a division of whole numbers. Do you agree or disagree? Explain.

16. Write each expression in simplest form by rationalizing the denominator.
  a) $\dfrac{5\sqrt{10}}{\sqrt{3}}$ **c)** $\dfrac{\sqrt{6} + \sqrt{2}}{\sqrt{6}}$
  b) $\dfrac{2\sqrt{2} - \sqrt{5}}{\sqrt{5}}$ **d)** $\dfrac{\sqrt{80} + 2\sqrt{3}}{3\sqrt{5}}$

17. Sound travels through different materials at different speeds. The speed of sound is modelled by

$$S = \sqrt{\frac{E}{d}}$$

where S represents the speed in metres per second, E represents the elasticity of the material in newtons per square metre (N/m^2), and d represents the density of the material in kilograms per cubic metre (kg/m^3). Determine the speed of sound through each material as a radical in simplest form.

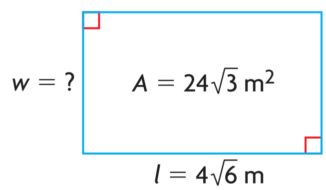
a) a material with $E = 4000$ N/m^2 and $d = 0.25$ kg/m^3

b) a material with $E = 320$ N/m^2 and $d = 0.20$ kg/m^3

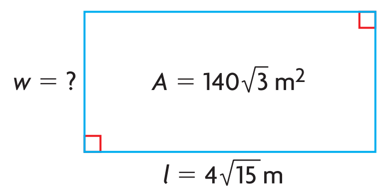
18. Buildings in snowy areas often have steep roofs. The steepness, or pitch, is expressed as the height of a roof divided by its width. Determine the pitch, in simplest form, for a building whose roof is $4\sqrt{3}$ m high and $2\sqrt{14}$ m wide.

19. Determine the width, w, of each rectangle.

a)



b)



20. Which expression has the lesser value? Explain how you know.

A. $2\sqrt{2}(9\sqrt{3} - \sqrt{12})$ **B.** $\dfrac{\sqrt{288} + 30\sqrt{2}}{2\sqrt{3}}$

21. a) Explain why you should keep numbers as radicals when calculating. Justify your response with an example.

b) Simplify $\dfrac{3\sqrt{8}}{\sqrt{6}} \cdot \dfrac{\sqrt{24}}{\sqrt{2}}$. Show your work, and explain what properties you used.

Closing

22. How is multiplying and dividing radicals like multiplying and dividing algebraic expressions? Explain using an example.

Extending

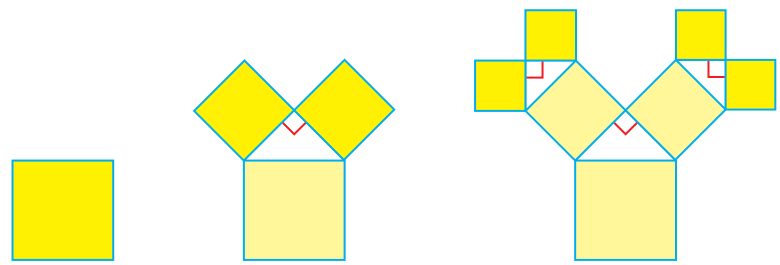
23. Morana rewrote $\dfrac{1}{\sqrt[3]{3}}$ with a rational denominator in the form $\dfrac{\sqrt[3]{9}}{3}$. Are these equivalent expressions? Explain.

Applying Problem-Solving Strategies

Defining a Fractal

A Pythagorean fractal tree starts at stage 1 with a square of side length 1 unit. At every consecutive stage, two squares are attached to the last square(s) drawn.

A. The first three stages of a Pythagorean fractal tree are shown. Determine the lengths of the sides of the smallest square at each stage.



B. What pattern exists between the measurements? Explain.

C. Draw two more stages and determine the side lengths of the added squares.

D. Determine the exact area of each of the first five stages of this Pythagorean fractal tree.

The Strategy

E. Explain the strategy you used to determine the measurements of the sides.

F. Explain the strategy you used to determine the area of each stage.

G. What pattern exists in the area measurements? Explain.

H. Describe how you would calculate the area of the 10th stage of the fractal tree.

YOU WILL NEED

- graph paper
- ruler

4 Mid-Chapter Review

FREQUENTLY ASKED Questions

Study Aid

- See Lesson 4.1.
- Try Mid-Chapter Review Questions 1, 2, 4, and 5.

Q: How do you compare numerical radicals?

A: To compare radicals, express them in their simplest form by expressing the radicand in terms of powers of its prime factors. Then estimate using the values of perfect squares.

For example: Which numerical radical is greater?

$$\begin{aligned}\sqrt{32} &= \sqrt{2^5}\\ &= \sqrt{2^4}\cdot\sqrt{2}\\ &= 2^2\sqrt{2}\\ &= 4\sqrt{2}\end{aligned}$$

$\sqrt{2}$ is close to 1.5,

so $4\sqrt{2}$ is about 6.

$$\begin{aligned}\sqrt[3]{72} &= \sqrt[3]{2\cdot 2\cdot 2\cdot 3\cdot 3}\\ &= \sqrt[3]{2^3}\cdot\sqrt[3]{3\cdot 3}\\ &= 2\sqrt[3]{9}\end{aligned}$$

9 is greater than 8 but close to it,

so $2\sqrt[3]{9}$ is about $2\cdot 2$ or 4.

$\sqrt{32}$ is greater than $\sqrt[3]{72}$.

Study Aid

- See Lesson 4.1.
- Try Mid-Chapter Review Questions 2, 4, and 5.

Q: How do you express a mixed radical as an entire radical?

A: Expand the mixed radical and express it as a multiplication statement of its factors, then determine the product of those factors.

For example:

$$\begin{aligned}-6\sqrt{5} &= -1\cdot\sqrt{6\cdot 6\cdot 5}\\ &= -\sqrt{180}\end{aligned}$$

$$\begin{aligned}7\sqrt[3]{2} &= \sqrt[3]{7\cdot 7\cdot 7\cdot 2}\\ &= \sqrt[3]{686}\end{aligned}$$

Study Aid

- See Lesson 4.2.
- Try Mid-Chapter Review Questions 6 to 9.

Q: How do you add and subtract radicals?

A: To add and subtract radicals, the radicals must have the same index and radicand.

For example:

$$\begin{aligned}12\sqrt{5}-\sqrt{180} &= 12\sqrt{5}-\sqrt{2^2\cdot 3^2}\cdot\sqrt{5}\\ &= 12\sqrt{5}-2\cdot 3\cdot\sqrt{5}\\ &= 12\sqrt{5}-6\sqrt{5}\\ &= 6\sqrt{5}\end{aligned}$$

Study Aid

- See Lesson 4.3.
- Try Mid-Chapter Review Questions 10 to 13.

Q: How do you multiply and divide radicals?

A: To multiply radicals, multiply the radicands. To divide radicals, divide the radicands and simplify. You can also rationalize the denominator and simplify.

For example:

$$\begin{aligned}\frac{12\sqrt{5}}{\sqrt{45}} &= \frac{12}{1}\sqrt{\frac{5}{45}}\\ &= 12\sqrt{\frac{1}{9}}\\ &= 12\cdot\frac{1}{3}\\ &= 4\end{aligned}$$

PRACTISING

Lesson 4.1

1. Estimate. Write your answers to the nearest tenth, if necessary.

a) $\sqrt{81}$
b) $\sqrt{249}$
c) $-\sqrt[3]{64}$
d) $-\sqrt{102}$
e) $\sqrt[3]{27}$
f) $\sqrt[3]{-125}$

2. Express as a mixed radical.

a) $\sqrt{32}$
b) $\sqrt{128}$
c) $-\sqrt[3]{532}$
d) $-\sqrt{54}$
e) $\sqrt[3]{108}$
f) $\sqrt[3]{-1024}$

3. Express as an entire radical.

a) $5\sqrt{3}$
b) $11\sqrt{4}$
c) $-4\sqrt[3]{216}$
d) $-4\sqrt{2}$
e) $2\sqrt[3]{81}$
f) $6\sqrt[3]{-8}$

4. Order from least to greatest:

$-\sqrt{101}, -2\sqrt[3]{8}, 4\sqrt[3]{-27}, -\sqrt{121}, -2\sqrt{25}$

5. Jackson is creating a triangular window for his cottage out of stained glass. One design has side lengths of 130 cm, 130 cm, and 200 cm. A second design has side lengths of 140 cm, 140 cm, and 140 cm. Which design will use more stained glass?

Lesson 4.2

6. Simplify.

a) $2\sqrt{3} + 4\sqrt{3} + \sqrt{3}$
b) $7\sqrt{2} + 9\sqrt{2} + 2\sqrt{2}$
c) $-2\sqrt{5} + 7\sqrt{5} + 3\sqrt{5}$
d) $-2\sqrt{7} + \sqrt{7} + 8\sqrt{7}$
e) $\sqrt{8} + 6\sqrt{8} + 5\sqrt{8}$

7. Simplify.

a) $\sqrt{6} - 2\sqrt{6} - \sqrt{6}$
b) $-\sqrt{4} - 5\sqrt{4} - 3\sqrt{4}$
c) $-2\sqrt{3} - 8\sqrt{3} - \sqrt{3}$
d) $\sqrt{10} - 2\sqrt{10} - 7\sqrt{10}$
e) $-2\sqrt{12} - \sqrt{12} - 4\sqrt{12}$

8. Simplify.

a) $\sqrt{75} + \sqrt{150}$
b) $\sqrt{81} + \sqrt{27} - \sqrt{49}$
c) $2\sqrt{7} + \sqrt{28} - \sqrt{63}$
d) $2\sqrt{98} - \sqrt{50}$
e) $2\sqrt{3} + \sqrt{108} - 5\sqrt{2}$

9. Tamlyn covered the top of a rectangular cake with 1200 cm^2 of chocolate fondant (a sheet of pliable icing). The length of the cake is twice the width of the cake. What are the dimensions of the top of the cake in radical form?

Lesson 4.3

10. Simplify. Express your answer in simplest form.

a) $\sqrt{7} \cdot \sqrt{8}$
b) $\sqrt{12} \cdot \sqrt{10}$
c) $3\sqrt{5} \cdot \sqrt{15}$
d) $-\sqrt{26} \cdot \sqrt{14} \cdot \sqrt{2}$
e) $-2\sqrt{25} \cdot -3\sqrt{10} \cdot -\sqrt{3}$

11. Simplify. Express your answer in simplest form.

a) $\dfrac{2\sqrt{10}}{\sqrt{5}}$
b) $\dfrac{12\sqrt{7}}{-2\sqrt{7}}$
c) $\dfrac{-13\sqrt{12}}{26\sqrt{6}}$
d) $\dfrac{28\sqrt{10}}{2\sqrt{2}}$
e) $\dfrac{27\sqrt{15}}{-9\sqrt{3}}$

12. Expand and simplify.

a) $\sqrt{2}(4 + 5\sqrt{3})$
b) $-7\sqrt{6}(6\sqrt{8} - 2)$
c) $(\sqrt{3} + \sqrt{7})(5 + 8\sqrt{10})$
d) $(2\sqrt{3} + 3\sqrt{5})(2\sqrt{3} - 3\sqrt{5})$

13. Raj used 14.5 m^3 of cement for his square patio. The height of the patio is 0.25 m. Determine the exact length and exact width of the patio.

4.4 Simplifying Algebraic Expressions Involving Radicals

GOAL

Simplify radical expressions that contain variable radicands.

EXPLORE...

- Choose two numbers that are opposites. Plot both numbers on a number line. How far is each of these numbers from zero? Is distance a positive or negative quantity? Repeat for several other pairs of opposite numbers. Compare results with other students and discuss what you found.

restrictions

The values of the variable in an expression that ensure the expression is defined.

LEARN ABOUT the Math

Algebraic expressions contain variables. Some algebraic expressions contain radicals, such as $\sqrt{x}$, $\sqrt{x^2}$, $\sqrt{x^3}$, and $\sqrt{x^4}$.

? Are radical expressions that involve variables defined for all real numbers, and is it possible to express them in simplest form?

EXAMPLE 1 Working with radicals that contain variables

For each expression above, explain any **restrictions** on the variable, then write the expression in its simplest form.

Melinda's Solution

a) $\sqrt{x}$ is defined when $x \geq 0$, where $x \in \mathrm{R}$.

All the radical expressions above involve the principal square root, because the square root sign indicates the positive square root. You cannot determine the square root of a negative number, so the expression $\sqrt{x}$ is defined only for real numbers greater than or equal to zero.

$\sqrt{x}$ cannot be simplified further.

I cannot express x as a power with an even exponent, so I cannot write this expression in simpler terms. x is the smallest possible radicand.

b) $\sqrt{x^2}$ is defined when $x \in \mathrm{R}$.

Since the variable is squared, the result will always be positive regardless of whether x is positive or negative. So, the square root can always be determined.

$\sqrt{x^2} = |x|$

The square root of x^2 must equal the **absolute value** of x.

Since I am determining the principal square root, the result is always positive, which is what the absolute value symbol ensures.

For example, if $x = -3$ and I want to determine $\sqrt{x^2}$, then by substitution I know that $\sqrt{(-3)^2} = \sqrt{9}$ and $\sqrt{9} = 3$. So, $\sqrt{(-3)^2} = 3$. If I had written $\sqrt{x^2} = x$, my answer could be -3, which would be incorrect.

c) $\sqrt{x^3}$ is defined when $x \geq 0$, where $x \in \mathrm{R}$.

When a positive number is cubed, the result is positive; but when a negative number is cubed, the result is negative. Since you cannot determine the square root of a negative number, the expression $\sqrt{x^3}$ is defined only for real numbers 0 or greater.

$$\sqrt{x^3} = \sqrt{x^2} \cdot \sqrt{x}$$
$$\sqrt{x^3} = |x| \cdot \sqrt{x}$$

I wrote x^3 as the product of two powers, one of which has an even exponent. When a power has even exponents it is a perfect square, so its square root can be determined by dividing the exponent by 2. I used the fact that $\sqrt{x^2} = |x|$ to simplify.

d) $\sqrt{x^4}$ is defined when $x \in \mathrm{R}$.

Since the variable is raised to an even exponent, the result will always be positive regardless whether x is positive or negative. So, this square root can always be determined.

$$\sqrt{x^4} = \sqrt{x^2} \cdot \sqrt{x^2}$$
$$\sqrt{x^4} = |x| \cdot |x|$$
$$\sqrt{x^4} = x^2$$

I wrote x^4 as the product of two powers, both of which have even exponents. I used the fact that $\sqrt{x^2} = |x|$ to simplify.

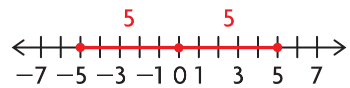
x^2 will always be positive, regardless of whether x is positive or negative.

absolute value

The distance of a number from 0 on a number line; the absolute value of x is denoted as

$$|x| = \begin{cases} x, \text{ If } x \geq 0 \\ -x, \text{ If } x < 0 \end{cases}$$

e.g., $|-5| = 5$



Both 5 and -5 are 5 units from 0.

Communication *Tip*

When working with radicands that contain a variable, the use of absolute value notation ensures that the principal square root is always represented. To simplify things, from this point forward in this resource, assume that x is the principal square root of x^2. In other words, $\sqrt{x^2} = x$.

Reflecting

A. Consider $\sqrt{x^n}$, where $n \in \mathrm{N}$. For what values of x is this expression defined? Explain.

B. Mark claims that $\sqrt{x^{2n}} = x^n$, where $n \in \mathrm{N}$ and $n > 1$. Is he correct? Explain.

C. José claims that $\sqrt[3]{x^3} = x$. Do you agree or disagree? Justify your decision.

APPLY the Math

EXAMPLE 2 Adding and subtracting algebraic expression involving radicals

State any restrictions on the variable, then simplify each expression.

a) $\sqrt{x} + 5\sqrt{x}$ **b)** $2\sqrt{4x^4} - \sqrt{8x^4}$

Bert's Solution

a) $\sqrt{x} + 5\sqrt{x}$ is defined when $x \geq 0$, where $x \in \mathrm{R}$.

$\sqrt{x}$ is defined only when x is either zero or positive.

$$1\sqrt{x} + 5\sqrt{x} = 6\sqrt{x}$$

These are like radicals. I simplified by adding the coefficients.

b) $2\sqrt{4x^4} - \sqrt{8x^4}$ is defined when $x \in \mathrm{R}$.

Both radicands will be greater than or equal to zero for any real number value of the variable.

$$2\sqrt{4x^4} - \sqrt{8x^4} = 2 \cdot \sqrt{2^2 \cdot x^4} - \sqrt{2^2 \cdot x^4} \cdot \sqrt{2}$$

To simplify, I factored each expression under the radicand using prime factors that had the highest even exponents possible.

$$= 2 \cdot 2x^2 - 2 \cdot x^2\sqrt{2}$$

$$= 4x^2 - 2x^2\sqrt{2}$$

I can't simplify further.

EXAMPLE 3 Simplifying algebraic expressions involving radicals

State any restrictions on the variable, then simplify each expression.

a) $4\sqrt{18x^3}$ **b)** $-7y^2\sqrt{8y^5}$ **c)** $\sqrt{x - 5}$

Meg's Solution

a) $4\sqrt{18x^3}$ is defined when $x \geq 0$, where $x \in \mathbf{R}$.

When a positive number is raised to an odd power, the result is positive; but when a negative number is raised to an odd power, the result is negative. Since you cannot determine the square root of a negative number, the expression is defined only for real numbers greater than or equal to zero.

$$4\sqrt{18x^3} = 4\sqrt{3^2 \cdot x^2} \cdot \sqrt{2x}$$

I wrote $18x^3$ as the product of two radicals. In the first radical, I chose factors that had the largest even exponents possible to create a perfect square.

$$= 4 \cdot 3 \cdot x \cdot \sqrt{2x}$$

$$= 12 \cdot x \cdot \sqrt{2x}$$

I multiplied the integers to simplify.

b) $-7y^2\sqrt{8y^5}$ is defined when $y \geq 0$, where $y \in \mathbf{R}$.

If y is negative, then the expression under the radical sign will be negative and undefined. The expression is defined only for real numbers greater than or equal to 0.

$$-7y^2\sqrt{8y^5} = -7y^2\sqrt{2^2 \cdot y^4} \cdot \sqrt{2y}$$

I wrote $8y^5$ as the product of two radicals. In the first radical, I chose factors that had the largest even exponents possible to create a perfect square.

$$= -7y^2 \cdot 2y^2 \cdot \sqrt{2y}$$

I used the fact that $\sqrt{y^4} = y^2$ to simplify.

$$= -14y^4\sqrt{2y}$$

I multiplied the integer coefficients and added the exponents on the variable to simplify.

c) $\sqrt{x - 5}$ is defined when $x - 5 \geq 0$. The expression is defined when $x \geq 5$, where $x \in \mathbf{R}$.

The square root sign tells me that $x - 5$ must be positive. This will occur when x is 5 or greater.

$\sqrt{x - 5}$ cannot be simplified.

$x - 5$ cannot be expressed using prime factors that involve even exponents.

Your Turn

Explain how you can manipulate a radical that contains both numbers and variables in its radicand to create a perfect square that can be used to simplify the expression. Use an example in your explanation.

EXAMPLE 4 Multiplying algebraic expressions involving radicals

State any restrictions on the variable, then simplify each expression.

a) $(5\sqrt{6x^2})(-2x\sqrt{2})$ **b)** $-3\sqrt{x}(2\sqrt{2} - 3x)$ **c)** $(2\sqrt{x} + 3)(5 - 3\sqrt{x})$

Chantelle's Solution

a) $(5\sqrt{6x^2})(-2x\sqrt{2x})$ is defined when $x \geq 0$, where $x \in \mathrm{R}$.

$6x^2$ will always be positive, since x is squared, so $\sqrt{6x^2}$ is defined for any value of x. But $\sqrt{2x}$ is defined only when $2x$ is positive. Therefore, x must be greater than or equal to zero to ensure both radicands will never be negative.

$$(5\sqrt{6x^2})(-2x\sqrt{2x}) = (5)(-2x) \cdot (\sqrt{6x^2})(\sqrt{2x})$$

I rearranged the terms so the radicals were beside each other, and multiplied.

$$= -10x \cdot \sqrt{12x^3}$$
$$= -10x \cdot \sqrt{2^2 \cdot x^2} \cdot \sqrt{3x}$$

I simplified the radical.

$$= -10x \cdot 2 \cdot x \cdot \sqrt{3x}$$

I multiplied again to simplify further.

$$= -20x^2\sqrt{3x}$$

b) $-3\sqrt{x}(2\sqrt{2} - 3x)$ is defined when $x \geq 0$, where $x \in \mathrm{R}$.

$\sqrt{x}$ is defined only when x is either 0 or positive. This means x must be greater than or equal to zero.

$$-3\sqrt{x}(2\sqrt{2} - 3x) = (-3\sqrt{x})(2\sqrt{2}) - (-3\sqrt{x})(3x)$$

I used the distributive property to expand. I multiplied the products in each term.

$$= -6\sqrt{2x} + 9x\sqrt{x}$$

I couldn't simplify any further because the expression didn't contain like radicals.

c) $(2\sqrt{x} + 3)(5 - 3\sqrt{x})$ is defined when $x \geq 0$, where $x \in \mathrm{R}$.

$\sqrt{x}$ is defined only when x is either 0 or positive. This means x must be greater than or equal to zero.

$$(2\sqrt{x} + 3)(5 - 3\sqrt{x})$$

I used the distributive property to expand.

$$= (2\sqrt{x})(5) + (2\sqrt{x})(-3\sqrt{x}) + (3)(5) + (3)(-3\sqrt{x})$$

$= 10\sqrt{x} - 6\sqrt{x^2} + 15 - 9\sqrt{x}$

$= 10\sqrt{x} - 6x + 15 - 9\sqrt{x}$

I simplified by subtracting like radicals.

$= -6x + \sqrt{x} + 15$

Your Turn

When do you need to use the distributive property to multiply expressions that contain radicals? Use examples in your explanation.

EXAMPLE 5 Simplifying algebraic expressions involving radicals and division

State any restrictions on the variable, then simplify each expression.

a) $\dfrac{15\sqrt{x^3}}{-3\sqrt{x^2}}$ **b)** $\dfrac{6\sqrt{5} - 2\sqrt{24x^3}}{2\sqrt{x}}$

Dwayne's Solution

a) $\dfrac{15\sqrt{x^3}}{-3\sqrt{x^2}}$ is defined for $x > 0$, where $x \in \mathrm{R}$.

$-3\sqrt{x^2}$ is defined for any value of x, because x^2 is always positive.

$15\sqrt{x^3}$ is defined for any positive value of x.

The entire expression is undefined when $x = 0$, because division by zero is undefined.

$= -5\sqrt{\dfrac{x^3}{x^2}}$

I divided the integers. The radicals in the numerator and denominator had the same index, so I combined them into one radical.

$= -5\sqrt{x}$

I divided the terms in the radicand by subtracting the exponents to write the expression in simplest form.

b) $\dfrac{6\sqrt{5} - 2\sqrt{24x^3}}{2\sqrt{x}}$ is defined for $x > 0$, where $x \in \mathrm{R}$.

$2\sqrt{x}$ is defined for any positive value of x.

$-2\sqrt{24x^3}$ is defined for any positive value of x.

The entire expression is undefined when $x = 0$, because division by zero is undefined.

$$= \frac{3\sqrt{5} - \sqrt{24x^3}}{\sqrt{x}}$$

I divided each integer in each radical by 2 in the numerator and the denominator.

$$= \frac{(3\sqrt{5} - \sqrt{24x^3})}{\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}}$$

I rationalized the denominator by multiplying by $\sqrt{x}$ in both the numerator and denominator.

$$= \frac{3\sqrt{5x} - \sqrt{24x^4}}{\sqrt{x^2}}$$

I used the distributive property to expand the numerator and multiplied the radicals in the denominator.

$$= \frac{3\sqrt{5x} - \sqrt{4x^4} \cdot \sqrt{6}}{x}$$

$$= \frac{3\sqrt{5x} - 2x^2\sqrt{6}}{x}$$

I simplified.

Your Turn

Create a rational algebraic expression that contains radicals that cannot be simplified by dividing. State the restrictions on the variables of your expression, then simplify it.

In Summary

Key Ideas

- When working with an algebraic expression involving radicals, it is important to state any restrictions on the variable; otherwise, the expression does not have meaning. For example, $\sqrt{x-2}$ is defined only when $x \geq 2, x \in \mathrm{R}$.
- The square root of all powers with an even exponent, such as $\sqrt{x^2}$, $\sqrt{x^4}$, and $\sqrt{x^6}$, is defined for all values of x; the square root of powers with an odd exponent, such as $\sqrt{x}$, $\sqrt{x^3}$, and $\sqrt{x^5}$, is defined for $x \geq 0, x \in \mathrm{R}$.
- The symbol $\sqrt{}$ indicates the principal square root, which is always positive.
- The square root of x^2 must equal the absolute value of x, denoted $|x|$, to ensure that the principal square root is represented.
- You simplify, add, subtract, multiply, and divide algebraic expressions with radicals using the same principles used for numerical expressions with radicals.

Need to Know

- You can add or subtract like radicals when they have the same index and radicand:
$$6\sqrt{x} + 2\sqrt{x} = 8\sqrt{x}$$
- You can use the following properties to help simplify algebraic radical expressions:
  - the commutative property: $5\sqrt{x} = \sqrt{x} \cdot 5$
  - the associative property: $\sqrt{x}(5\sqrt{2}) = 5(\sqrt{x} \cdot \sqrt{2})$ or $\sqrt{2}\,(5\sqrt{x})$
  - the distributive property: $2(\sqrt{x} + 1) = 2\sqrt{x} + 2 \cdot 1$
  - the multiplicative identity property: $\dfrac{2}{\sqrt{x}} = \dfrac{2}{\sqrt{x}} \cdot \dfrac{\sqrt{x}}{\sqrt{x}}$

CHECK Your Understanding

1. State any restrictions on the variable in each expression. Explain how you decided.

a) $4\sqrt{x^6}$ **c)** $\sqrt{x+3}$

b) $2\sqrt{x^3}$ **d)** $\dfrac{\sqrt{x}}{x}$

2. State any restrictions on the variable, then simplify.

a) $2\sqrt{45x^4}$ **c)** $2\sqrt{12x^3}$

b) $\sqrt{9x^2}$ **d)** $-3x\sqrt{8x^5}$

3. State any restrictions on the variable(s), then simplify.
   a) $15\sqrt{2x} - 7\sqrt{2x} - \sqrt{2x}$
   b) $36\sqrt{3x^3} - 10\sqrt{3x^3} + 28\sqrt{3x^3}$
   c) $(5\sqrt{x})(2\sqrt{x})$
   d) $\dfrac{9\sqrt{x^5}}{-3\sqrt{x}}$

PRACTISING

4. Simplify.
   a) $(5x^3\sqrt{x})(\sqrt{2x^3})$
   b) $(-5\sqrt{x})(2\sqrt{8x^3})$
   c) $\dfrac{-36\sqrt{x^3}}{12\sqrt{x}}$
   d) $\dfrac{(3\sqrt{x})(4\sqrt{x^3})}{6\sqrt{x^4}}$

Communication | ***Tip***

When no restrictions on the variables are stated in a problem, assume that the variables are of the set of real numbers, R.

5. Charlene simplified $7x\sqrt{x} + 3\sqrt{x^3}$ as shown, but some of her writing was smudged.
   a) Determine the missing elements in each step.
   b) Explain what Charlene did in each step.

	$7x\sqrt{x} + 3\sqrt{x^3}$
Step 1:	$7x\sqrt{x} + 3 \cdot \sqrt{\blacksquare} \cdot \sqrt{x}$
Step 2:	$7x\sqrt{x} + 3 \cdot \blacksquare \cdot \sqrt{x}$
Step 3:	$\sqrt{x}\,(\blacksquare + \blacksquare)$

6. Simplify.
   a) $\sqrt{12x} - 5\sqrt{3x} + \sqrt{27x}$
   b) $(2x\sqrt{x})(\sqrt{x} + 4\sqrt{x^5})$
   c) $\dfrac{3 - x}{\sqrt{x}}$
   d) $\dfrac{4\sqrt{x^3}}{\sqrt{8x}}$

7. Explain how dividing radicals and rationalizing radical expressions are the same and how they are different. Provide an example that supports your reasoning.

8. Simplify.
   a) $(5\sqrt{x})(2\sqrt{2x})$
   b) $\sqrt{2x}\,(\sqrt{2x} + 4x)$
   c) $(-3\sqrt{x})(\sqrt{x^3} - 4x)$
   d) $(\sqrt{x} + 2)(\sqrt{x} + 5)$

9. Simplify.
   a) $\sqrt{4x} + 2\sqrt{16x}$
   b) $4\sqrt{x^4} - 2\sqrt{x^4}$
   c) $3\sqrt{y}(4 - 2\sqrt{y^3})$
   d) $(5 - \sqrt{y})^2$

10. Simplify.

a) $\dfrac{\sqrt{x^7}}{\sqrt{x}}$ b) $\dfrac{\sqrt{8x^3}}{\sqrt{2x}}$ c) $\dfrac{\sqrt{50x^4}}{\sqrt{2x^2}}$ d) $\dfrac{6\sqrt{x^4}}{3x^2}$

11. State the restrictions on the variable in each expression.

a) $\sqrt{x-9}$ c) $\sqrt{3x+6}$

b) $\sqrt{x+4}$ d) $\sqrt{3x-2}$

12. What would you multiply each quotient by in order to rationalize the denominator of each expression?

a) $\dfrac{\sqrt{x}}{\sqrt{5}}$ b) $\dfrac{-3}{\sqrt{x}}$ c) $\dfrac{12}{\sqrt{7x}}$ d) $\dfrac{5}{2\sqrt{x}}$

13. Explain how to rationalize the denominator of $\dfrac{3+2\sqrt{x}}{4\sqrt{x}}$.

14. At Trent's local mall, a circular water fountain is set inside a triangular prism, as shown. Trent has learned that this inscribed circle is related to the sides of this supporting triangle by the formula

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

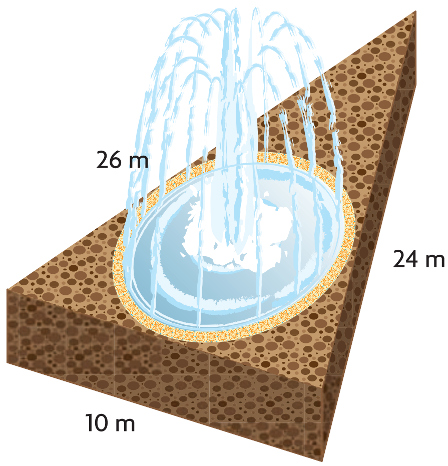
where r represents the radius of the inscribed circle in metres, s is half the perimeter of the triangle, and a, b, and c represent the lengths of the sides of the triangle.

He would like to determine the radius of the fountain.

a) Trent feels that he can divide each s in the numerator with the s in the denominator. Is he correct? Explain.

b) What should Trent do first to simplify this expression?

c) What are the restrictions on s in this context? Explain.



15. State the restrictions on the variable, then simplify the expression:

$$\frac{(3\sqrt{2}-2x)(3\sqrt{2}+2x)}{2\sqrt{x}}$$

Closing

16. How is multiplying and dividing algebraic expressions with radicals like multiplying and dividing numerical expressions with radical values? Explain, using an example.

Extending

17. The lateral surface area for a paper cup, S, is defined by

$$S = \pi r \cdot \sqrt{r^2 + h^2}$$

where r represents the radius and h represents the height. Melanie simplified this as shown to the right. Is she correct? Justify your answer.

Melanie's Solution

$S = \pi r\sqrt{r^2 + h^2}$

$S = \pi r(\sqrt{r^2} + \sqrt{h^2})$

$S = \pi r(r + h)$

$S = \pi r^2 + \pi rh$

4.5 Exploring Radical Equations

YOU WILL NEED

- variety of different-sized balls
- string, rulers, tape measure
- calculator

GOAL

Develop a strategy for solving radical equations.

EXPLORE the Math

Alex is in charge of organizing a table tennis tournament. During a break in the tournament, he began to wonder about the surface area and volume of the table tennis balls. He determined both values using a ruler, a piece of string, and these formulas for the radius of a ball, r:

$$r = \sqrt{\frac{A}{4\pi}}$$

where A represents the surface area of the ball

$$r = \sqrt[3]{\frac{3V}{4\pi}}$$

where V represents the volume of the ball

? How can you use these materials and formulas to determine the volume and surface area of any spherical ball?

Reflecting

A. How did you determine the radius of your ball? What difficulties did you have, if any?

B. How did you determine the surface area and volume of your ball using the given formulas? How was the strategy you used the same and how was it different for each situation?

C. Sam says you can determine the surface area of a sphere using $A = 4\pi r^2$ and its volume using $V = \frac{4}{3}\pi r^3$. Calculate the surface area and volume of your ball using these formulas. What do you notice?

D. How are the formulas in part C related to those Alex used? Explain.

Communication *Tip*

Use the π key on your calculator. Round your answer only at the end of your calculations.

In Summary

Key Idea

- You can solve equations involving square roots and cube roots using inverse operations.

Need to Know

- Squaring a number, a^2, is the inverse operation of taking a square root, $\sqrt{a}$.
- Cubing a number, a^3, is the inverse operation of taking a cube root, $\sqrt[3]{a}$.

FURTHER Your Understanding

1. Solve each equation.

a) $5\sqrt{x} = 35$

b) $\sqrt{\frac{x + 3}{2}} = 4$

c) $\sqrt[3]{\frac{4x}{5}} = 2$

d) $\sqrt[3]{x - 1} = 3$

2. Ella installs lawn watering systems. The radius of sprayed water, r, in metres, can be expressed as

$$r = \sqrt{0.64\,A}$$

where A represents the area watered, in square metres.

Ella has set each sprayer to spray at a radius of 1.2 m. Determine the area of grass watered by one sprayer, to the nearest tenth of a square metre.

3. The speed of a tsunami, S, in kilometres per hour, can be modelled by

$$S = 356\sqrt{d}$$

where d represents the average depth of the water, in kilometres. Determine the average depth of the water, to the nearest hundredth of a kilometre, for a tsunami travelling at 100 km/h.

4. The radius of a cylindrical tank, r, in metres, is given by

$$r = \sqrt{\frac{V}{1.5\pi}}$$

where V represents the volume of water the tank holds, in cubic metres. Determine the volume of water in a tank with a radius of 0.9 m, to the nearest hundredth of a cubic metre.

5. The square root of the sum of twice a number and 5 is 7.

a) Create the radical equation defined by the above statement.

b) Solve your radical equation.

c) Repeat parts a) and b) if "square root" in the statement is changed to "cube root."

4.6 Solving Radical Equations

YOU WILL NEED

- calculator

GOAL

Solve and verify radical equations that contain a single radical.

EXPLORE…

- Marnie claims that the equation $\sqrt{(2x-1)^2} = x$ has two solutions: $x = 1$ and $x = \frac{1}{3}$. Do you agree or disagree? Justify your decision.

LEARN ABOUT the Math

In the previous lesson, you solved radical equations by using inverse operations. The equations you used all had a single solution. Consider these equations:

A. $\sqrt{3x} = 6$

B. $\sqrt{x+2} = -3$

C. $\sqrt{x-1} + 3 = 4$

D. $\sqrt[3]{2x} = 4$

? Does a radical equation always have a solution?

EXAMPLE 1 Using inverse operations to solve radical equations

Solve each of the radical equations above. Explain your solution.

Melvin's Solution

Equation A: $3x \geq 0$

$x \geq 0$

$\sqrt{3x} = 6$ is defined for $x \geq 0$, where $x \in R$.

The radicand in this equation must be greater than or equal to zero, since its index is 2, indicating a square root. I solved the inequality to determine the restrictions on *x*.

$$\sqrt{3x} = 6$$
$$(\sqrt{3x})^2 = (6)^2$$
$$3x = 36$$
$$x = 12$$

The radical is isolated, so I squared both sides of the equation to eliminate the radical. This resulted in a linear equation I could solve.

Verify:

$\sqrt{3x} = 6$

$x = 12$

LS	RS
$\sqrt{3(12)}$	6
$\sqrt{36}$	
6	

This equation has one solution, $x = 12$.

I verified my solution by substituting 12 for x in the original equation. Since the left side and right side are equal, $x = 12$ is the solution.

Equation B: $x + 2 \geq 0$

$x \geq -2$

$\sqrt{x+2} = -3$ is defined for $x \geq -2$, where $x \in \mathbf{R}$.

The radicand in this equation must be greater than or equal to zero, since its index is 2, indicating a square root. I solved the inequality to determine the restrictions on x.

$$\begin{aligned} \sqrt{x+2} &= -3 \\ (\sqrt{x+2})^2 &= (-3)^2 \\ x + 2 &= 9 \\ x &= 7 \end{aligned}$$

The radical is isolated, so I squared both sides of the equation to eliminate the radical then solved the resulting equation.

Verify:

$\sqrt{x+2} = -3$

$x = 7$

LS	RS
$\sqrt{7+2}$	-3
$\sqrt{9}$	
3	

This equation has no solution.

When $\sqrt{x+2}$ was squared, -3 was also squared, resulting in the equation $(\sqrt{x+2})^2 = (-3)^2$

But you get the same equation when you square both sides when solving

$$\begin{aligned} \sqrt{x+2} &= 3 \\ (\sqrt{x+2})^2 &= 3^2 \end{aligned}$$

$x = 7$ is the solution to this equation.

I verified my solution by substituting 7 for x in the original equation. Since the left side and right side are not equal, $x = 7$ is not a solution. In this situation, 7 is an **extraneous root**.

extraneous root
A root that does not satisfy the initial conditions that were introduced while solving an equation. Root is another word for solution.

Equation C: $x - 1 \geq 0$

$x \geq 1$

$\sqrt{x - 1} + 3 = 4$ is defined for $x \geq 1$, where $x \in \mathbf{R}$.

The radicand in this equation must be greater than or equal to 0, since its index is 2, indicating a square root. I solved the inequality to determine the restriction on *x*.

$$\sqrt{x - 1} + 3 = 4$$
$$\sqrt{x - 1} = 1$$

I isolated the radical by subtracting 3 from both sides of the equation.

$$(\sqrt{x - 1})^2 = 1^2$$
$$x - 1 = 1$$
$$x = 2$$

I squared both sides to eliminate the radical.

Verify:

$\sqrt{x - 1} + 3 = 4$

$x = 2$

LS	RS
$\sqrt{2 - 1} + 3$	4
$\sqrt{1} + 3$	
$1 + 3$	
4	

I verified the solution by substituting 2 for *x* in the original equation.

This equation has one solution, $x = 2$.

Equation D: $\sqrt[3]{2x} = 4$ is defined for $x \in \mathbf{R}$.

The radicand in this equation can be either negative, zero, or positive, since its index is 3, indicating a cube root. This means that the equation is defined for all real numbers.

$$\sqrt[3]{2x} = 4$$
$$(\sqrt[3]{2x})^3 = (4)^3$$
$$2x = 64$$
$$x = 32$$

The radical is isolated, so I cubed both sides to eliminate the radical.

Verify:

$\sqrt[3]{2x} = 4$

$x = 32$

LS	RS
$\sqrt[3]{2(32)}$	4
$\sqrt[3]{64}$	
4	

I checked my solution by substituting 4 for *x* in the original equation. Since the left side and right side are equal, $x = 4$ is the solution.

This equation has one solution, $x = 4$.

Reflecting

A. Jim says he could tell that equation B had no solution by inspection. Do you agree? Explain.

B. How do you eliminate a radical while solving an equation?

C. Explain why you should always verify the solutions of a radical equation.

APPLY the Math

EXAMPLE 2 Modelling a situation with a radical equation

The forward and backward motion of a swing can be modelled using the formula

$$T = 2\pi\sqrt{\frac{L}{9.8}}$$

where T represents the time in seconds for a swing to return to its original position, and L represents the length of the chain supporting the swing, in metres. When Cara was swinging, it took 2.5 s for the swing to return to its original position. Determine the length of the chain supporting her swing to the nearest centimetre.

Ramesh's Solution: Isolating the variable then substituting

The expression on the right of the formula is defined for $L \geq 0$, where $L \in \mathrm{R}$.

$$T = 2\pi\sqrt{\frac{L}{9.8}}$$

$$\frac{T}{2\pi} = \sqrt{\frac{L}{9.8}}$$

I needed to isolate L. However, I needed to start by isolating the radical. To do so, I divided both sides of the equation by 2π.

$$\left(\frac{T}{2\pi}\right)^2 = \left(\sqrt{\frac{L}{9.8}}\right)^2$$

$$\left(\frac{T}{2\pi}\right)^2 = \frac{L}{9.8}$$

I squared both sides of the equation to eliminate the radical.

$$9.8\left(\frac{T}{2\pi}\right)^2 = L$$

I multiplied both sides by 9.8 to isolate L.

$$9.8\left(\frac{2.5}{2\pi}\right)^2 = L$$

I substituted the given information into the equation.

$$9.8(0.397\ldots)^2 = L$$
$$9.8(0.158\ldots) = L$$
$$1.551\ldots = L$$

The length of the swing's chain is 1.55 m, which is 155 cm.

Nicola's Solution: Substituting then isolating the variable

The expression on the right of the formula is defined for $L \geq 0$, where $L \in \mathrm{R}$.

$$T = 2\pi\sqrt{\frac{L}{9.8}}$$

$$\frac{2.5}{2\pi} = \sqrt{\frac{L}{9.8}}$$

I substituted the given information into the formula.

$$\left(\frac{2.5}{2\pi}\right)^2 = \left(\sqrt{\frac{L}{9.8}}\right)^2$$

$$\left(\frac{2.5}{2\pi}\right)^2 = \frac{L}{9.8}$$

I squared both sides of the equation to eliminate the radical.

$$9.8\left(\frac{2.5}{2\pi}\right)^2 = L$$

I multiplied both sides by 9.8 to isolate L.

$$9.8\left(\frac{2.5}{2\pi}\right)^2 = L$$

$$9.8(0.397\ldots)^2 = L$$

$$9.8(0.158\ldots) = L$$

$$1.551\ldots = L$$

I evaluated the numerical expression on the left side of the equation.

The length of the swing's chain is 1.55 m or 155 cm.

Your Turn

Verify the solution.

In Summary

Key Idea

- When you solve an equation with a variable in the radical, first isolate the radical on one side of the equation. Then square both sides of the equation if the radical is a square root. Cube both sides if the radical is a cube root. Then solve for the variable as you would normally.

Need to Know

- When you solve an equation with a variable in the radical,
  - you need to restrict the variable to ensure that the radicand of a square root is not negative; however, the radicand of a cube root may be negative
  - squaring a radical may introduce an invalid solution, called an extraneous root; for this reason, you need to verify each solution by substituting into the original equation

CHECK *Your Understanding*

1. State any restrictions on x, then solve each equation.

a) $\sqrt{x} = 4$

b) $\sqrt{x} = 6$

c) $\sqrt{x + 1} = 2$

d) $\sqrt{x + 3} = 4$

2. State any restrictions on x, then solve each equation.

a) $\sqrt[3]{x} = -3$

b) $\sqrt{2x} = 5$

c) $\sqrt[3]{x + 4} = -2$

d) $\sqrt{2x + 4} = 6$

3. Wendy said that there are no solutions for the equation

$$\sqrt{2x} = -4$$

Is she correct? Explain how you know.

PRACTISING

4. State any restrictions on x, then solve each equation.

a) $\sqrt{4x} = 4$

b) $\sqrt[3]{8x} = 2$

5. Describe the first step you would take to solve each equation. Justify your decision.

a) $\sqrt{x} = 7$

b) $\sqrt{x + 5} = 12$

c) $\sqrt[3]{x - 3} = 4$

d) $8 = \sqrt{2x + 7} - 1$

6. State any restrictions on x, then solve each equation.

a) $\sqrt{x - 3} = 5$

b) $\sqrt[3]{4x + 7} = 3$

c) $2\sqrt{5x + 3} = 11$

d) $\frac{1}{2}\sqrt{3x - 2} = 4$

7. There are many equations related to electrical engineering. For instance, the voltage of an electrical device, V, is related to the power used, P, in watts (W), and resistance, R, in ohms (Ω), by the equation

$$V = \sqrt{P \cdot R}$$

Determine the amount of power needed to run a device that requires a voltage of 120 V and that contains a 2 Ω resistor.

8. State any restrictions on x, then solve each equation.

a) $\sqrt{2x + 17} = 5$

b) $\sqrt[3]{6 - 2x} + 1 = -1$

c) $\sqrt{2(5x + 3)} = -4$

d) $\sqrt{33 - 6x} + 4 = 13$

9. Create a radical equation that results in an extraneous root. Verify that your equation has no solution.

10. Bob claims that $-\sqrt{4x + 1} = -5$ and $\sqrt{4x + 1} = 5$ have the same solution, but $\sqrt{4x + 1} = -5$ has no solution. Do you agree or disagree? Explain.

11. A space station needs to rotate to create the illusion of gravity. A formula for determining the rotation rate to reproduce Earth's gravity is

$$N = \frac{42}{\pi}\sqrt{\frac{5}{r}}$$

where N represents the number of revolutions per minute and r represents the radius of the station in metres.

A station rotates 6.7 times per minute, producing an effect on the interior wall equivalent to Earth's gravity. Determine the radius of the space station.

12. Some collectors view comics as an investment. The effective rate of interest, r, earned by an investment can be defined by the formula

$$r = \sqrt[n]{\frac{A}{P}} - 1$$

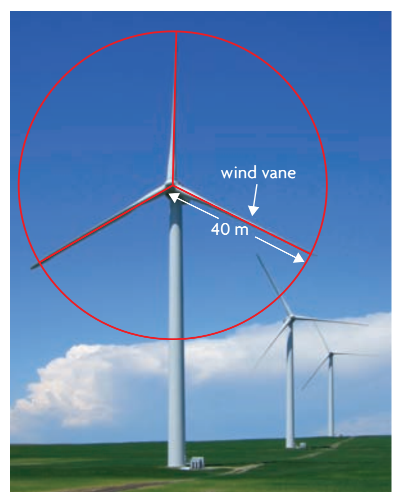
where P represents the initial investment, in dollars, that grows to a value of A dollars after n years.

Determine the initial price of a rare comic book that resold for \$1139 after two years, earning its owner 18% interest.

13. The amount of energy, P, in watts (W), that a wind turbine with vanes 40 m long generates, is related to the wind speed, S, by the formula

$$S = \sqrt[3]{\frac{2P}{5026.5D}}$$

where D represents the density of the air where the turbine operates. If a wind turbine is built in an area in which the air density is 0.9 kg/m^3 and the average wind speed is 8 m/s, determine how much power this turbine can generate.



14. Melinda bought a circular table for her deck. When she placed it on the deck, she concluded that its area was about a quarter of the area of her 5 m by 5 m square deck. Determine the radius of the table to two decimal places. Justify your response.

15. Solve $\sqrt{20x + 50} + 3 = 11$. State the restrictions on x.

Closing

16. Create a radical equation and explain what steps are needed to solve it.

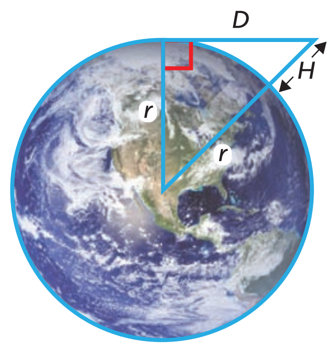
Extending

17. Solve $\sqrt[4]{x + 5} + 1 = 4$. Explain each step and state any restrictions on the variable.

Math in Action

How Far Is the Horizon?

If you have ever seen a ship appear on the horizon, you saw the top of the ship first, then the rest of the ship. This gradual appearance of the ship is owing to Earth's curvature. You can determine the sightline distance, D, using the radius of Earth, r, the height of the eyes of the observer above the surface, H, and the Pythagorean theorem.



- Use a basketball as a model of Earth.
- Tape to the basketball a small object, such as a pencil sharpener, to represent you.
- Use a ruler to determine the measures of D and H as these relate to your object.
- Use these measures to estimate the radius of the basketball.
- Compare your results with those of other groups.

4 Chapter Self-Test

1. Express each radical as a mixed radical in simplest form.

a) $\sqrt{1176}$ **b)** $-\sqrt{896}$ **c)** $\sqrt[3]{1296}$ **d)** $\sqrt[3]{-2560}$

2. Order the numbers $12\sqrt{2}$, $3\sqrt{12}$, $\sqrt{121}$, $4\sqrt{5}$, $\sqrt[3]{1000}$ from least to greatest without using a calculator. Explain what you did.

3. Simplify each expression. Explain each step.

a) $5\sqrt{3} + 4\sqrt{2} - \sqrt{3} - 2\sqrt{2}$

b) $\sqrt{275} + \sqrt{27} - \sqrt{363} - \sqrt{176}$

4. Express in simplest form.

a) $(2\sqrt{10})(-3\sqrt{5})$

b) $-3\sqrt{6}(\sqrt{18} - 2\sqrt{5})$

c) $(2 + 3\sqrt{x})^2$

5. Simplify.

a) $\dfrac{3\sqrt{80}}{2\sqrt{4}}$ **b)** $\dfrac{3\sqrt{7}}{\sqrt{5}}$ **c)** $\dfrac{9\sqrt{x^5}}{3\sqrt{x}}$

6. State the restrictions on the variables, then simplify.

a) $x\sqrt{72x^4}$ **b)** $(2x\sqrt{x^2})(-3x\sqrt{x^3})$

7. Kshawn has been contracted to water lawns for the summer using circular sprayers. The radius of the sprayed water, r, in metres, is modelled by

$$r = \sqrt{1.6A}$$

where A represents the area of grass watered in square metres. Kshawn has set a sprayer to spray a radius of 1.5 m. Determine the area of grass watered by the sprayer, to the nearest tenth of a square metre.

8. Explain why you need to consider restrictions when working with radical expressions. Provide an example of a monomial radicand and a binomial radicand and state the restrictions for both examples.

9. a) State the restrictions on x, then solve:

$$\sqrt{16x + 20} - 3 = -1$$

b) Is the root extraneous? Explain.

WHAT DO You Think Now? Revisit **What Do You Think?** on page 175. How have your answers and explanations changed?

4 Chapter Review

FREQUENTLY ASKED Questions

Study Aid

- See Lesson 4.4, Examples 1, 2, 3, 4, and 5.
- Try Chapter Review Question 7.

Q: How can you simplify a radical expression with variable radicands?

A1: You can factor the radicand and express the factors as powers, just as you would a numerical radicand.

For example:

$6\sqrt{80x^3}$

Since the radical sign indicates a square root, the expression is defined for $x \geq 0$, where $x \in R$.

$= 6\sqrt{2^4 \cdot 5 \cdot x^2 \cdot x}$

$= 6 \cdot 2^2 \cdot x\sqrt{5x}$

$= 24x\sqrt{5x}$

A2: You can multiply or divide radicals with the same index.

For example:

$\dfrac{18\sqrt{x^3}}{-6\sqrt{x^2}}$

$= \dfrac{18}{-6}\sqrt{\dfrac{x^3}{x^2}}$

$= -3\sqrt{x}$

Since division by zero is not allowed, then $x \neq 0$. Since the radical sign over x^3 indicates a square root, then the expression is defined for $x > 0$, where $x \in R$.

A3: You can rationalize an expression with a radical in the denominator by multiplying the numerator and denominator by the same radical expression.

For example:

$\dfrac{2\sqrt{14}}{3\sqrt{x}}$

Since division by zero is not allowed, then $x \neq 0$. Since the radical sign indicates a square root, the expression is defined for $x > 0$, where $x \in R$.

$= \dfrac{2\sqrt{14}}{3\sqrt{x}} \cdot \dfrac{3\sqrt{x}}{3\sqrt{x}}$

$= \dfrac{2\sqrt{14} \cdot \sqrt{x}}{3\sqrt{x} \cdot \sqrt{x}}$

$= \dfrac{2\sqrt{14x}}{3x}$

It is not possible to divide, because there are no common factors. So, multiplying by 1, in the form $\dfrac{\sqrt{x}}{\sqrt{x}}$, allows the expression to be simplified.

Q: **How can you solve an equation with a variable in the radical?**

A: First state the restrictions on the variable. Then isolate the radical and either square both sides of the equation (if the radical is a square root) or cube both sides (if the radical is a cube root). Then solve for the variable as you would normally. Lastly, substitute each solution into the original equation to ensure it is not extraneous.

For example:

a) $\sqrt{2x + 5} - 1 = 4$

Since $\sqrt{2x + 5}$ cannot be negative, then $2x + 5 \geq 0$, or $x \geq -2.5$.

The restrictions are $x \geq -2.5$, where $x \in \mathrm{R}$.

$$\sqrt{2x + 5} = 4 + 1$$

Isolate the radical by adding 1 to both sides.

$$\begin{aligned}(\sqrt{2x + 5})^2 &= (5)^2\\ 2x + 5 &= 25\\ 2x &= 20\\ x &= 10\end{aligned}$$

Square both sides.

Verify:

$\sqrt{2x + 5} - 1 = 4$

$x = 10$

LS	RS
$\sqrt{2(10) + 5} - 1$	4
$\sqrt{25} - 1$	
4	

The solution $x = 2$ is valid.

b) $\sqrt{4x} = -6$

Since $4x$ cannot be negative, then $4x \geq 0$, or $x \geq 0$.

This equation has no solution. Squaring both sides and solving for x will lead to an extraneous root.

The left side of the equation indicates the principal square root, which is positive. This means the left side can never equal the right side, since the right side is negative.

Study | *Aid*

- See Lesson 4.6, Examples 1 and 2.
- Try Chapter Review Questions 8 to 11.

PRACTISING

Lesson 4.1

1. Express each radical as a mixed radical in simplest form.

a) $\sqrt{72}$ c) $\sqrt{40}$
b) $\sqrt{600}$ d) $\sqrt[3]{250}$

2. Express each mixed radical as an entire radical.

a) $6\sqrt{5}$ c) $4\sqrt{14}$
b) $12\sqrt{7}$ d) $-3\sqrt[3]{4}$

Lesson 4.2

3. Simplify.

a) $\sqrt{36} + \sqrt{42}$
b) $6\sqrt{2} + 3\sqrt{48} + \sqrt{96}$
c) $4\sqrt{104} - 6\sqrt{2} - \sqrt{242}$
d) $6\sqrt{36} - 6\sqrt{48} - 5\sqrt{216}$

Lesson 4.3

4. Simplify.

a) $\sqrt{24} \cdot \sqrt{42}$
b) $3\sqrt{25} \cdot 5\sqrt{10}$
c) $-2\sqrt{30} \cdot 2\sqrt{40}$
d) $-2\sqrt{14} \cdot -2\sqrt{42}$

5. Simplify.

a) $6(4 + \sqrt{12})$
b) $\sqrt{5}(2 - \sqrt{15})$
c) $\sqrt{6}(\sqrt{20} - 8\sqrt{6})$
d) $(5\sqrt{2} - \sqrt{3})(2\sqrt{2} + 4\sqrt{3})$

6. Tom was asked to write $\sqrt{12}$ in simplest form. He wrote:

$$\begin{aligned}\sqrt{12} &= \sqrt{2 + 2 + 2 + 2 + 2 + 2}\\ &= \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2}\\ &= 6\sqrt{2}\end{aligned}$$

What was Tom's error?

Lesson 4.4

7. State any restrictions on the variable(s), then simplify.

a) $(4x^3\sqrt{x})(\sqrt{2x^3})$ c) $\dfrac{-128\sqrt{x^3}}{6\sqrt{x}}$
b) $\dfrac{-18\sqrt{8x^3}}{9\sqrt{2x}}$ d) $\dfrac{2\sqrt{5x}}{3\sqrt{6x}}$

Lesson 4.5

8. Police can use skid marks to determine how fast a vehicle was travelling. The speed, s, in kilometres per hour, is related to the length of the skid mark, d, in metres, and the coefficient of friction of the road, f, by this formula:

$$s = \sqrt{252df}$$

Determine the length of a skid mark made by a car travelling at 80 km/h on a concrete road with a coefficient of friction measuring 0.76.

Lesson 4.6

9. State the restrictions on x, then solve each equation.

a) $\sqrt{x} = 11$ c) $3\sqrt{7x} + 3 = 12$
b) $\sqrt{x + 3} = 14$ d) $\dfrac{1}{4}\sqrt{5x - 2} = 6$

10. State the restrictions on x, then solve each equation.

a) $\sqrt{3x + 16} = -5$
b) $\sqrt[3]{7 - 2x} + 1 = -3$
c) $\sqrt{4(3x + 12)} = 6$
d) $\sqrt{22 - 6x} + 4 = 2$

11. Jenny solved $\sqrt{x - 4} - 1 = 2$ as follows:

Step 1: $\sqrt{x - 4} = 2 - 1$

Step 2: $x - 4 = 1^2$

Step 3: $x = 1 + 4$

Step 4: $x = 5$

a) What are the restrictions on the variable? Justify your decision.
b) Where did Jenny make her first error? Explain.

4 Chapter Task

Designing a Radical Math Game

To celebrate Numeracy Week, the math club at Sonora's school has decided to hold a series of competitive math games during lunch hour. Groups of club members each chose a topic and created a game.

? What game can you create that demands skill in working with numerical and algebraic radicals?

A. Create a game for two or four players in which the players must answer questions. Players must demonstrate their ability to
- express entire radicals as mixed radicals in simplest form
- express mixed radicals as entire radicals
- simplify radical expressions that involve the operations addition, subtraction, multiplication, and division
- solve radical equations
- state restrictions on variables

B. Exchange games with another group and play each other's game. The creators should observe while their game is being played. After you have played both games, discuss what worked well and what needed improvement. Revise your game, based on the feedback.

C. Write a report that describes how the game is played and its rules. Include in your report all the questions and answers needed for the game. Discuss how you revised your game.

Task Checklist

- ✔ Are the rules of your game written clearly?
- ✔ Did you include answers to all the questions?
- ✔ Is your explanation of how and why you revised your game clear?

4

Project Connection

Carrying Out Your Research

As you continue with your project, you will need to conduct research and collect data. The strategies that follow will help you collect data.

Considering the Type of Data You Need

There are two different types of data that you need to consider: primary and secondary. Primary data is data that you collect yourself using surveys, interviews, and direct observations. Secondary data is data you obtain through other sources, such as online publications, journals, magazines, and newspapers.

Both primary data and secondary data have their pros and cons. Primary data provides specific information about your research question or statement, but may take time to collect and process. Secondary data is usually easier to obtain and can be analyzed in less time. However, because the data was gathered for other purposes, you may need to sift through it to find what you are looking for.

The type of data you choose can depend on many factors, including the research question, your skills, and available time and resources. Based on these and other factors, you may choose to use primary data, secondary data, or both.

Assessing the Reliability of Sources

When collecting primary data, you must ensure the following:

- For surveys, the sample size must be reasonably large and the random sampling technique must be well designed.
- For surveys or interviews, the questionnaires must be designed to avoid bias.
- For experiments or studies, the data must be free from measurement bias.
- The data must be compiled accurately.

When obtaining secondary data, you must ensure that the source of your data is reliable:

- If the data is from a report, determine what the author's credentials are, how up-to-date the data is, and whether other researchers have cited the same data.
- Be aware that data collection is often funded by an organization with an interest in the outcome or with an agenda that it is trying to further. When presenting the data, the authors may give higher priority to the interests of the organization than to the public interest. Knowing which organization has funded the data collection may help you decide how reliable the data is, or what type of bias may have influenced the collection or presentation of the data.
- If the data is from the Internet, check it against the following criteria:
  - Authority: The credentials of the author are provided and can be checked. Ideally, there should be a way to contact the author with questions.
  - Accuracy: The domain of the web address may help you determine the accuracy of the data. For example, web documents from academic sources (domain .edu), non-profit organizations and associations (domains .org and .net) and government departments (domains such as .gov and .ca) may have undergone vetting for accuracy before being published on the Internet.
  - Currency: When pages on a site are updated regularly and links are valid, the information is probably being actively managed. This could mean that the data is being checked and revised appropriately.

Accessing Resources

To gather secondary data, explore a variety of resources:

- textbooks
- scientific and historical journals and other expert publications
- newsgroups and discussion groups
- library databases, such as Electric Library Canada, which is a database of books, newspapers, magazines, and television and radio transcripts

People may be willing to help you with your research, perhaps by providing information they have or by pointing you to sources of information. Your school or community librarian can help you locate relevant sources, as can the librarians of local community colleges or universities. Other people, such as teachers, your parents or guardians, local professionals, and Elders and Knowledge Keepers may have valuable input. (Be sure to respect local community protocols when approaching Elders or Knowledge Keepers.) The only way to find out if someone can and will help is to ask. Make a list of people who might be able to help you obtain the information you need, and then identify how you might contact each person on your list.

PROJECT EXAMPLE — Carrying out your research

Sarah chose, "Which Western province or territory grew the fastest over the last century, and why?" as her research question. She has decided to use 1900 to 2000 as the time period. How can she find relevant data?

Sarah's Search

Since my question involves a historical event over a wide area, I decided to rely on secondary data. I started my search using the Internet. I did a search for "provincial populations Canada 1900 to 2000" and found many websites. I had to look at quite a few until I found the following link:

[PDF] History resources from Statistics **Canada**
File Format: PDF/Adobe Acrobat - Quick View
Population of **Canada**, by **province**, census dates, 1851 to 1976. A125-163. ... series reviews conditions in **Canada** from **1900 to 2000**. Articles include: ...
www.cshc.ubc.ca/TC_Smith.pdf - Similar

This led me to a document from the University of British Columbia that cited a document from Statistics Canada, based on census data, that showed the provincial populations from 1851 to 1976. I went to the Statistics Canada website and searched for the census data, but I couldn't find it. So I tried another general search, "historical statistics Canada population," and found the link below:

Historical statistics of Canada: Section A: Population and Migration
Oct 22, 2008 ... Table A1 **Estimated population of Canada**, 1867 to 1977. ..... and **Migration**", in the **Historical Statistics of Canada**, first edition, p. ...
www.statcan.gc.ca › Home › Sections - Cached - Similar

This led me to data I was looking for:

Table A2-14
Population of Canada, by province, census dates, 1851 to 1976

Source: for 1851 to 1951, Statistics Canada (formerly Dominion Bureau of Statistics), *Census of Canada, 1951*, vol. X, table 1; for 1956, *Census of Canada, 1956*, vol. I, table 1; for 1961 *Census of Canada, 1961*, Vol. I, part 1, table 12, (Catalogue 92-536); for 1966, *Census of Canada, 1966*, vol. I, table 14, (Catalogue 92-608); for 1971, *Census of Canada, 1971*, vol. I, part 2, table 14, (Catalogue 92-716); for 1976, *Census of Canada, 1976*, vol. II, table 11, (Catalogue 92-824).

For a brief discussion of possible under-enumeration in earlier censuses, 1851, 1861 and 1871, see first edition of this volume, pp. 3-4. For completeness of enumeration in censuses of 1961 to 1976, see series A15-53 below.

I now have some data I can use. I feel confident that the data is authoritative and accurate, because I believe that the source is reliable. I will continue looking for more current data, from 1976 to 2000. I will also need to search for information about reasons for population changes during this time period. I will ask the school librarian to help me look for other sources.

Your Turn

A. Decide if you will use primary data, secondary data, or both. Explain how you made your decision.

B. Make a plan you can follow to collect your data.

C. Carry out your plan to collect your data. Make sure that you record your successful searches, so you can easily access these sources at a later time. You should also record detailed information about your sources, so you can cite them in your report. See your teacher for the preferred format for endnotes, footnotes, or in-text citations.